Unified Science Theory

(an overview)

Unified Science Theory

Organization – Goals – Progress

Organization

Management – Basic Research Company

Publications – Basic Research Press

Principal Funding Source – Book Mart Corp

Goals

Obtain Unlimited Useful Energy

Solve Human Aging

Improve Human Relations

Progress

Useful Energy – The Neutrino model in *The Grand Unified Theory of Physics*

Aging – *The Chemistry and Mechanics of Human Aging*

Human Relations – A Possible Mechanism of Decision Making

– The Human Soul

– Language (Including Phonetics)

– The Encyclopedia of Unified Physics

UnifiedScienceTheory.com

Preamble

This website, Unifiedsciencetheory.com, reports the work-in-progress of the organization Basic Research Company.

Basic Research Company was founded to work on the three most significant problems facing humanity. The three problems are:

1. How do we get all the useful energy we want?
2. How do we cure the most significant disease mankind has ever experienced, i.e., aging?
3. How do we keep humans from murdering other humans?

In this website we show what progress has been made on solving each one of these problems. First, we summarize the problems and then we list the progress toward solving each.

We developed a unification of all the basic observations in the world of physics and reported the results in the book, *The Grand Unified Theory of Physics*. This theory shows from where all the useful energy in the universe comes. When we think of energy we think of chemical and nuclear energy. We show in the *Grand Unified Theory of Physics* that all of these sources ultimately are produced by gravitation, see the *Human Cosmos*. Gravitation, in turn, is produced by neutrinos, see the *Grand Unified Theory of Physics*. In turn, the neutrinos organize particles from an ether-gas universe to extract useful energy from a uniform gas. This theory is very detailed in the grand unified theory book. Current experimental research should be directed toward modeling the neutrino. If someone can duplicate on a large scale what nature does on a small scale then they should be able to extract useful energy from the atmospheric gas (and instigate global cooling).

More people die from age-related effects than from any other cause. Throughout the last half century great progress has been made toward understanding the biological cell, but very little progress has been made on understanding aging. We recently proposed a theory of aging which shows some promise of being the actual cause. This theory is presented in preliminary draft form in *The Chemistry and Mechanics of Human Aging*. Basically this draft outlines a (long range) research program to verify the theory and to seek a cure. Obtaining a cure for aging is anticipated to be a very difficult problem.

There has always been significant turmoil in societies throughout history and today's society has shown no improvement. Tens of millions of people have been killed recently by oppressive governments killing their own citizens (to wit China and the Soviet Union). Tens of millions of people have been killed by nations fighting nations in wars which have almost continually been waged during the last century. As a result of technological advances small groups of people can kill groups of people which are remotely located (to wit the attack by Asians on the United

States of America on September 11, 2001). Finally there are people killing their neighbors. There are other types of social conflicts, of course.

We have a general plan for attacking social problems but we have even less confidence in the plan's success than in our theory of aging. However we do have a plan and we're working on it. First, we want to see a large cadre of teachers developed who understand physical science such as that exemplified by the *Encyclopedia of Unified Physics*, see our website physicsunifiedtheory.com. This body of knowledge also gives a basic understanding of language and mathematics and thus provides a starting basis for all people to understand a significant amount about each other. Next we need a good understanding of chemistry and biology – which we feel is currently available. We do offer the theory of aging though as an area of research which, if successful, could greatly reduce human suffering. In the area of social sciences we have just begun. We have outlined a theory of decision making, see "A Possible Mechanism of Decision-Making." We outline the major non-physical things people experience in the "Human Soul." Whatever those non-physical things are or whatever the soul is, we know that it controls almost all actions by humans.

As everyone can see we have set forth some lofty goals in the above paragraphs. It may take many years of effort by our successors to make significant progress toward achieving these goals. We feel, however, that stating the goals and outlining approaches will expedite solutions.

Unified Science Theory

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1. Introduction

In 1938, some 70 years ago, a group of philosophers including Rudolph Carnap, Charles Morris, and Otto Neurath published a two volume set of books devoted to unified science. The books are: *International Encyclopedia of Unified Science*, Volumes 1 and 2; edited by Otto Neurath, Rudolph Carnap, and Charles Morris; published by The University of Chicago Press, Chicago, Illinois, 1938. These books discussed science with an organization quite similar to the organization here as illustrated by their contents which follow:

International Encyclopedia of Unified Science
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One would assume that the two volumes were their “first cut” at developing an encyclopedia. However, no additional books were produced.

In this international encyclopedia mention was made of the French *Encyclopedie*, written in the 18th century. This was “a giant work achieved by co-operation of a great many specialists.” This French encyclopedia might be similar to the German Encyclopedia of Science produced in recent years.

The unified science theory presented here differs from others in that certain "physical building blocks" are postulated (i.e., the basic particle, and its characteristics) then language "blocks" are postulated to explain these physical building blocks and their activities. After these postulates are presented then the structure of language, mathematics, and mechanics are developed to produce a unified theory of physics. The applied science of chemistry and then the biological sciences and social sciences follow naturally.

2. Foundations of Language

Preface

This book *Foundations of Language* sets forth the ultimate foundations of a unified theory of science. If, and when, an encyclopedia of unified science were completed this book would be volume number one.

This book begins with a presentation of written signs which correspond to the human physiological configurations utilized for making the sounds (i.e., verbal signs) of language. These configurations consist of lungs (exhalation of air), voice box (glottis, vibrating or non-vibrating), the air channels (pharynx, mouth, and nose), the uvula and velum, tongue, hard palate, teeth, and lips. The time phased locations of the moving parts of the voice systems produce the articulation which results in the various sounds (symbols) of language. Part A (Phonetics) of the book presents the verbal and written signs of language.

Part B (Semantics) of the book, presents the correspondence of the signs of language with their "meaning." One of the most basic meanings is the correspondence of positive integers to the "manyness" of physical objects. For example the number 2 names the manyness of pairs. The fundamental tenets of set theory, logic, mathematics, geometry, and physics are presented in part B.

Part A (Phonetics) shows how to position the mouth, lips, and tongue and how to actuate the vocal chords to produce the sounds identified by the International Phonetic Association as those most used for human communication. The symbols used by the English language are listed first. This is followed by the symbols used in Chinese, Portuguese, Spanish, Russian, then Arabic. The next section gives the alphabets used by the six different languages along with words illustrating the use of the sounds. For Chinese the recently introduced pinyin alphabet is used. This is followed by translations into the five languages of the words used in Part B of this book. These translations should provide the majority of the literate people in the world with the ability to pronounce the words in Part B.

The *Foundations of Language* text lays out the basic foundations of language, and as it turns out, the foundations of mathematics and the foundations of physics also. The material is written in such a manner that any intelligent person from anywhere on the earth should be able to read and understand the text.

The first semantics section of the text presents the most basic conventions of English grammar such as reading a book from front to back and reading a page left to right and top to bottom. The text begins by naming quantities and the quantities initially are just quantities of balls. We further restrict the text initially to dealing with the ten digits. The logic then is developed for discourse on quantities up to nine balls. With this language machinery we then develop the positive integers. Next we develop the basis for the postulates of geometry. We show physically how to develop the concept of a straight line using balls and the mathematical foundations developed in the preceding chapters. The concepts of points, (flat) planes, and solids are developed. Finally, the concepts of mass, displacement, time, velocity, and changes of velocity are developed.

In previous works, *Principles of Science,* Joseph M. Brown, ISBN 0-962676-0-2, Basic Research Press, Starkville, MS, 1991 and *Fundamentals of Physics,* Joseph M. Brown, ISBN 0-9626768-1-0, Basic Research Press, Starkville, MS 1999 show, in outline form, how the foundations of language and mathematics couples to the foundations of mechanics.

The number of balls is "mass" (where a proportionality constant is used), and volumes are described by number of ball diameter lengths, widths, and heights. The amazing thing about the language for describing numbers of balls and the development of mathematics is accomplished by using (actually postulating) just a few simple words to name objects and to name operations for combining and separating objects.

We next note the utter simplicity of language and its immediate and direct connection with physical entities. It is almost inconceivable that the language and resulting mathematics could be "inconsistent." It would seem that the physics of kinetic particles would have to have inconsistencies in order for the language of mathematics presented here to have inconsistencies.

Notwithstanding all the positive things we have said (and believe) in the above 9 paragraphs, there are deficiencies in the development. The ultimate goal is to select a minimum number of postulate words and then define all new words in terms of these postulated words. We have not done that. However, we have indicated how it can be done. We leave that as a project for some future researcher.

FOUNDATIONS OF LANGUAGE

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3. Arithmetic

Arithmetic is used for counting the number (or amount) of things. In the book *Foundations of Language*, the number 1 is introduced to name the manyness of items where the manyness can have a varying integral value and the integral value (or quantity) is the minimum amount but it is some. The number 2 was introduced to name the quantity resulting when the quantity 1 is combined with another quantity of 1. Extending this combining process the positive integers are developed.

All of the concepts of arithmetic are developed rigorously from the material developed in *Foundations of Language*. In the part of science known as "arithmetic," techniques (i.e., algorithms) are evolved to efficiently manipulate quantities – for example, to add two quantities. Techniques are also introduced for extending the positive integers to negative integers and to rational numbers.

Subtraction is the opposite of addition. The number 10 plus 6 is 16. The number 16 less (or minus) 6 is 10. We have the number 6 and we want to subtract 10. We write 6 – 10 and we obtain -4, or 6 – 10 = -4. The number -4 is an integer and it is negative, it is a negative integer. Introducing negative integers and zero permits the subtraction of any integer from another.

Techniques are developed for adding a number to itself a multiple number of times, i.e., multiplication. The result of multiplication is a "product" (of two numbers). The two numbers multiplied together to obtain a "product" are called "factors."

The reverse operation, beginning with a product and one factor and then determining the other factor is called "division." If the product is 8 (for example) and the given factor is 2 the reverse operation is signified by 8/2 and the unknown factor is 4. If the product is 13 and the factor is 3 the reverse operation is signified by 13/3 and there is no integer which can be multiplied by 3 to give the product 13. What is done is to add to the integers quantities of the form 13/3, or any integer (on top) "divided by" any other integer (on bottom). The "on top" number is the "numerator" an the "on bottom" number is the "denominator." In some cases the factor to be determined requires an unlimited number for the numerator as well as an unlimited number for denominator. Such a factor (or number) is called an "irrational" number while if the numerator and denominator are finite they are called rational numbers.

The concept of powers is the result of multiplying a number by itself one, or more times. For example, 10^3 is short for 10 x 10 x 10, or 10^3 = 10 x 10 x 10 and we say 10 to the power 3 is 10 x 10 x 10, and sometimes we say the base 10 to the power 3 is 10 x 10 x 10 = 1000. What about the reverse problem where we have the number 1000 and we ask what power must be used to obtain 1000 if the base is 10. For this problem we introduce the logarithm where the logarithm of a number is the power to which it is necessary to raise the base to produce the given number. Thus the logarithm for the base 10 to give 1000 is 3, or in shorter form $\log_{10} 1000 = 3$. Consider a more difficult problem $10^{?} = 896$, where we want to know what power 10 must be raised to obtain 896. The answer of course is somewhere between 2 and 3. We write the problem as $\log_{10} 896 = ?$. Tables have been prepared which the logarithm of the number can be found. In this case $\log_{10} 896 = 2.9523$, or $10^{2.9523} = 896$.

Computation is simplified by using decimals rather than rational numbers. For example 1/5 can be represented by 2/10 which, using the decimal notation, is written as .2 (or 0.2) which simply means 2/10 (or two tenths). As another example 3/8 can be written as 3x125/8x125 = 375/1000 = .375 (or 0.375) which is named "point three seven five."

In arithmetic techniques are developed for adding, subtracting, multiplying, and dividing all kinds of rational as well as irrational numbers, decimals, logarithms, etc. Arithmetic is extremely useful in almost all areas of life. Almost everyone uses arithmetic many times daily throughout their lifetime after childhood.

The book *Arithmetic: An Introduction to Mathematics* by L. Clark Lay, Macmillan Company, New York 1961 presents the material discussed in the above paragraphs.

4. Algebra

Algebra is a generalization of arithmetic. The "number" of arithmetic is represented by a symbol which is a variable. In algebra we say x is a number and x can be any number defined in arithmetic. In algebra we say "let x be a number" or "let 'a' be a number and 'b' be a number (where 'b' may be a number different from 'a' or 'b' may represent the same number 'a')." Also, we can let x be a special number such as a positive integer or a real number greater than 2 and less than 6, for example.

Once we introduce into mathematics the concept of a symbol denoting a variable number we can start describing properties of numbers. A list of properties of numbers from which all the results (or theorems) of numbers can be derived is called axioms for the theory.

One list of axioms for the real numbers follows:

1. For any real numbers a and b we have $a + b = b + a$, which is the commutative law of addition.

2. For any real numbers a and b we have $ab = ba$, which is the commutative property of multiplication.

3. For any real numbers a, b, and c we have $(a + b) + c = a + (b + c)$, which is the associative law of addition.

4. For any real numbers a, b, and c we have $(ab)c = a(bc)$, which is the associative law of multiplication.

5. For any real numbers a, b, and c we have $a(b + c) = ab + ac$, which is the distributive property of multiplication.

6. For any real number a we have $a + 0 = 0 + a = a$, which is the addative identity law (0 is the addative identity number).

7. For any real number a we have $1 \times a = a \times 1 = a$, which is the multiplicative identity law (1 is the multiplicative identity number).

8. For any real number a there is a unique number $-a$ such that $a + (-a) = -a + a = 0$, which is the additive inverse law.

9. For any real number a there is a unique number $1/a$ such that $a \times (1/a) = (1/a) \times a = 1$, which is the multiplicative inverse law.

10. For any real number a, $0 \times a = a \times 0 = 0$, which is the multiplicative property of zero.

The above ten statements are properties of the real number system. They are obvious results of the way the numbers were defined and they are just ten of the many properties of the real number system. However, this particular list of properties is special. All the properties of the real numbers can be derived from these ten properties. Such a set of properties is called a set of axioms (for the real number system). These axioms provide practical means of computation and generally handling real numbers.

We now ask if there is a simpler axiom system for real numbers rather than the above 10 axioms. It has been discovered that using a system of 5 axioms all the properties of the positive integers can be derived. By defining inverse addition (subtraction) the negative integers and zero are introduced, and by defining inverse multiplication the rational (and irrational) numbers are introduced to complete the real number system. These axioms were derived by Richard Dedekind in 1888 and by Guiseppe Peano in 1889. They are usually called the Peano postulates. We construct the set of positive integers P by the following rules:

1. 1 is a member of P

(Thus we have at least one number in the set of positive integers).

2. If a is a positive integer then the successor of a denoted $s(a)$, also is a positive integer.

(This means we have at least two members of the set. However, the successor of a, where a is 1 can not be just any quantity. $s(a)$ can not be 1 x 1 it has to be a quantity double the quantity 1. $s(a)$ must be 1 + 1.)

3. If a and b are positive integers and $s(a) = s(b)$ then $a = b$.

(This prevents a “loop” which could make the set of positive integers be finite. For example, without this postulate $s(1)$ could be 2 and $s(2)$ could be 1. There then would only be two positive integers.)

4. If $s(a)$ is in P then $s(a) \neq 1$.

(This makes 1 be the first positive integer. These postulates put an unlimited number of integers in the set and they are “ordered.”)

5. If Q is a subset of P, then $Q = P$ provided that 1 is in Q and if a is in Q, then $s(a)$ is in Q.

(This postulate prevents any elements other than 1, 2, 3, 4, …. Being in the set.)

All of these axioms are obvious results of the way the positive integers are defined in *Foundations of Language*.

Edmund Landau in *Foundations of Analyses* published by AMS Chelsea Publishing, Providence, Rhode Island 1956 begins with these 5 axioms and derives as theorems the set of 10 axioms for the real number system listed earlier.

The last axiom above is the axiom of mathematical induction, even though it doesn't look like the axiom of induction with which most of us are familiar. The familiar statement is:

> Let there be assigned to every positive integer n a statement S_n which may be true or false. Suppose S_1 is true and that for each n the statement S_{n+1} is true when the statements S_1, S_2, S_3, …., S_n are true. Then all S_n are true.

We will now derive the above theorem from axiom 5. In this derivation we use the results on pages 33, 37, 38, and 43 of George D. Mostow, Joseph H Sampson, and Jean-Pierre Meyer's *Fundamental Structures of Algebra*, McGraw-Hill Book Company, Inc. NY 1963.

First we define the system of integers as the set consisting of the positive integers, zero, and the negative integers and we denote this set as I. Let P be the set of positive integers and P' the set of negative integers. If a, b, are two elements of I then a is smaller than b (or b is greater than a) if $b - a$ is in I and we write $a < b$, or $b > a$.

Mostow proves the following three theorems:

1. If x is in P then either $x = 1$ or $x > 1$, i.e. 1 is the smallest positive element of I.

2. If $x < y$ then $x + 1 \leq y$, i.e., there is no element of I between x and $x + 1$.

3. Let M be any subset of P containing at least one element. Then M contains a smallest element, i.e., in M there is a unique element y such that y < z for any other element of M.

We can now prove the familiar form of the axiom of induction:

Theorem 4. Let there be assigned to every positive integer n a statement S_n which may be true or false. Suppose S_1 is true and for each n the statement S_{n+1} is true whenever S_1, S_2, S_3, …, S_n are true. Then all S_n are true. (This is a theorem only if S_n is a precise mathematical statement.)

Proof. Let F be the set of all positive integers n for which S_n is false. By theorem 3 the set F has a least element $n.$ If it is not empty, and in that case $n_0 > 1$ because S_1 has been assumed true. Then $n = n_{0\text{-}1}$ is a positive integer, and by the definition of F, all statements S_1, S_2, S_3, …, S_N must be true. Therefore, by assumption, $S_{n+1} = S_{n+0}$ is also true – which is a contradiction. Thus, F must be empty.

We now present a statement which is proven true using mathematical induction. For all positive integers n we want to prove

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

Let $P_n = 1 + 3 + 5 + \ldots + (2n - 1) = n^2$

Now for $n = 1$ we have $P_1 = 1 = 1^2$

Assuming $P(k)$ is true we have $1 + 3 + 5 + \ldots + (2k - 1) = k^2$

Now is P_{k+1} true? $1 + 3 + 5 + \ldots + (2k - 1) + [2(k + 1) - 1] \stackrel{?}{=} (k + 1)^2$

Or $1 + 3 + 5 + \ldots + (2k - 1) + 2(k + 1) - 1 \stackrel{?}{=} (k + 1)^2$

$$1 + 3 + 5 + \ldots + 4k = k^2 + 2k + 1$$

$$1 + 3 + 5 + \ldots + (2k - 1) = k^2$$

Thus we have P_{k+1} is true if $P(k)$ is true. So P_1 is true and P_{k+1} is true if P_k is true. By mathematical induction the initial equation is true for all positive integers.

Textbooks on algebra present topics in algebra which have practical application. Algebra is used several orders of magnitude less often than arithmetic. However there are practical problems which can only be solved using algebraic methods. Most often these are the "word" problems in algebra books. The topics in the remaining paragraphs are covered in a typical college algebra text book, see the book *College Algebra* by Claude Irwin Palmer and Wilson Lee Miser, 2nd ed., McGraw-Hill Book Company, Inc., NY 1937.

An algebraic equation (or inequality) with the unknown variable always raised to the first power is a linear equation (or inequality).

Methods for manipulating and solving linear equations with one variable, two variables, and many variables are presented. The use of determinants and matrices are introduced for handling systems of equations with many variables.

The quadratic equation has one variable raised to the second power and, generally, to the first power as well as to the zero power (i.e., a constant). Techniques for solving the quadratic equation are presented. The roots (i.e., solutions) of some quadratic equations appear with the form $a \pm \sqrt{-b}$, where a and b are real numbers and b is positive. There are practical uses for such solutions. What is done is to extend the real number system by adding complex numbers. We re-write $a \pm \sqrt{-b}$ as $a \pm \sqrt{-1}\cdot\sqrt{b}$, introduce imaginary unit $i = \sqrt{-1}$, define ic (= $i\sqrt{b}$) as an imaginary number, and $a + ic$ as a complex number. In case "c" is zero the complex number becomes a real number. Thus complex numbers envelop real numbers. With this invention all quadratic equations have solutions (which are complex numbers).

A polynomial is an algebraic equation with two or more terms. Usually the polynomial with two terms is called a binomial. A binomial to any power greater than unity has more than two terms and it is a polynomial, of course.

Let x and y be numbers (real or complex) and let n be a positive integer. Then the binomial theorem gives the equation for the expansion

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{1*2}x^{n-1}y^2 + \frac{n(n-1)(n-2)}{1*2*3}x^{n-3}b^3$$

$$+\cdots+ \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}x^{n-r+2}y^{r-1}$$

$$+ \frac{n(n-1)\cdots(n-r+1)}{r!}x^{n-r}y^r + \cdots + y^n$$

This result also is valid where n is any real number. This theorem is proven by using the axiom of mathematical induction.

An algebraic expression is integral if all the exponents of the variable are positive and rational if all the exponents are integers. A rational integral function of x of the n^{th} degree, i.e., a polynomial in x, is of the form

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$

Where n is a positive integer, $a_0 \neq 0$, and a_0, a_1, …, a_n are real numbers. A rational integral equation of the n^{th} degree is formed by equating the general polynomial $f(x)$ to zero. Thus

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

Or $f(x) = 0$

A rational integral equation is called an algebraic equation. If $n = 2$ the equation is the quadratic equation, if $n = 3$ it is a cubic, if $n = 4$, a biquadratic or quartic equation.

A root of an algebraic equation $f(x)=0$ is a value of x which when substituted into the left member gives zero (i.e., makes the equation balance).

The fundamental theorem of algebra is that every rational integral equation of the n^{th} degree, $f(x)=0$, has exactly n roots (each one of which is a real or complex number).

Techniques for solving equations with exponents larger than unity (powers) and smaller than unity (roots) are presented. Techniques for using logarithms also are presented. Processes which grow exponentially, such as money receiving interest which is retained by the lender and thus producing compound interest, utilize logarithms having a base of e = 2.7182818 …. .

Other subjects covered in algebra are such subjects as infinite sequences (an unlimited list of terms) and infinite series (a sum of an unlimited number of terms) and probability theory.

5. Advanced Algebra

In the last section "Algebra" we saw that starting with the real number system and introducing the concept of variable (i.e., letting a symbol represent more than one number) the algebra with which we are most familiar was developed. The subject of algebra is much broader than would be suspected from what we have presented. We have presented the algebra of the (real and complex) number system. If we begin with the mathematics of set theory, where sets have been evolved in the *Foundations of Language*, we see a vast array of algebraic structures (one of which, of course, is the algebra of the number system). Set theory also is used to illuminate concepts developed in the algebra of the number system.

Chapter VI of the book *An Introduction to Modern Mathematics*, by Albert Monjallon Published by John Wiley, 1967, extends the logic material in *Foundations of Language* to further illuminate the language roots of mathematics. Chapters I through V begin with the set theory foundations given in *Foundations of Language* and develop set theory, the concept of mathematical relations (like 4 is less than 6), and the fundamental concept of functions (a function relates an element of one set to an element of another set).

This book develops a set-theoretic basis for the algebra discussed in the section just before this one. The material of Monjallon also gives background material which is useful for understanding the broad structure of algebra with its many branches.

The book *An Introduction to Abstract Mathematical Systems* by David M. Burton, published by Addison-Wesley, Reading, Massachusetts, 1965, shows how group theory, field theory, and vector/matrix theory are developed from set theory. The complex number system is a field. Thus, this book shows how ordinary algebra fits into the broad structure of algebra. This book also shows the rigorous derivation starting with set theory, of vector and matrix theory. The one short-coming of this book from the practical needs of the unified science theory, is that the derivation of tensors was not included.

Vectors, tensors, and matrices are utilized in the rigorous derivation of the structure of physical science. Thus, these two books (by Monjallon and Burton) give the rigorous connection between the *Foundations of Language* and the beginning mathematics for physical science.

6. A Prelude to Geometry

Geometry deals with points, lines, planes, and solids. All of these concepts have been derived from basic postulates in *Foundations of Language*. The derivations there begin with a metric (i.e., a postulated measure of distance which is the diameter of the basic particle making up everything in the universe) and, of course, a three-dimensional space. Euclidean geometry, the geometry taught in secondary schools, deals primarily with relations between geometric entities. But it is not taught as a pure non-metric subject since number lines often are introduced. In Chapter VI of the book *Foundations of Modern Mathematics* by Lyle E. Mehlenbacher, published by Prindle, Weber & Schmidt, Incorporated, 1967, several simple finite geometries are discussed. These geometries give insight into what geometry is all about.

The book by Mehlenbacher also has chapters on mathematics, language, logic, other algebras, mathematical relations, foundations of analytic geometry, and the circular functions (i.e., trigonometry). Trigonometry is a small subset of analytic geometry.

7. Plane Geometry

The book *Basic Geometry* by George David Birkhoff and Ralph Beatley, 3rd edition published by Chelsea Publishing Company, New York, 1959, presents plane geometry. The presentation uses the following five axioms:

1. The points on any straight line can be numbered so that the number differences measure distances.
2. There is only one straight line through two points.
3. All half-lines having the same end-point can be numbered so that number differences measure angles.
4. All straight angles have the same measure.
5. Two geometric figures [polygons] are similar if all corresponding angles are equal and all corresponding distances [lengths of sides] are proportional.

This is a metric approach to plane geometry (to wit axioms 1 and 3). Many plane geometry books use the approach of relations between geometric entities and down-play the use of numerical measure in their presentation. It would appear that the metric emphasis would make the material easier to learn.

8. Solid Geometry

The book *Solid Geometry* by A. M. Welchons, W. R. Krickenberger, and Helen R. Pearson, published by Ginn and Company, New York, 1959, presents a conventional approach extending plane geometry to three dimensions to give solid geometry.

9. Plane Trigonometry

Trigonometry means triangle measurement. Trigonometry is a restricted form of analytic (i.e., algebraic) geometry. The starting point for trigonometry is to define functions which are the ratios of the lengths of two sides of a right triangle. Since there are three sides (say *a*, *b*, and *c*) there are 3 ratios with their reciprocals, to make the six ratios. Letting θ be one of the non-90° angles, *a* be the length of the side opposite θ, *b* the length of the side adjacent to θ, and *c* be the hypotenuse we have the six functions as:

$$\sin\theta = a/c = 1/\csc\theta, \quad \cos\theta = b/c = \sec\theta, \quad \tan\theta = a/b = 1/\cot\theta$$

Trigonometry is presented in *Plane Trigonometry* fourth edition by Fred W. Sparks and Paul K. Rees, published by Prentice-Hall, Inc. Englewood Cliffs, NJ, 1960.

10. Analytic Geometry

In analytic geometry all the various geometric entities are represented by numbers (or algebraic symbols). Theorems then are proven numerically instead of by geometric relations. *Essentials of Analytic Geometry* by David Raymond Curtiss and Elton James Moulton, published by D. C. Heath and Company, Boston, 1947, presents this subject. The book covers graphing, the line, the circle, conic sections, handling second degree equations, plane curves, tangents for plane geometry, and similar subjects for solid geometry.

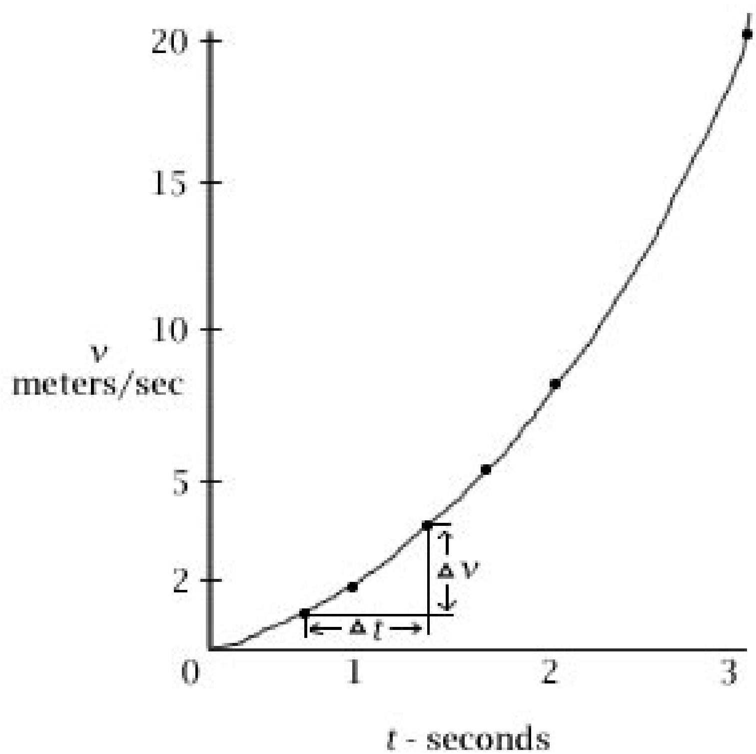
11. Introductory Matrices and Vectors

Matrices are used for solving systems of *n* linear algebraic equations with *n* unknowns. Vectors are used in physical science to represent displacements, velocities, and accelerations. The book *An Introduction to Matrices, Vectors, and Linear Programming* by Hugh G. Campbell, published by Appleton-Century-Crofts, New York, NY, 1965, is an elementary treatment of matrices and vectors.

This book extends and illuminates the theory presented in the last chapter of the book *An Introduction to Abstract Mathematical Systems* by David M. Burton, 1965.

12. Differential and Integral Calculus

One of the significant problems solved by the differential and integral calculus which pervades all of physical science is the determination of velocities when accelerations are given, displacements when velocities are given, and the inverse of these two situations. For example, if the velocity v of a particle is given in terms of time t by $v = kt^2$ we can graph the velocity versus time as shown below when $k = 2$.



The acceleration of the particle is given by the time rate of change of velocity. The time rate of change of velocity, i.e., the acceleration a, at t = 1 sec is approximated by $a = \Delta v / \Delta t = 3.2 / 0.7 = 4.6$ meters/sec^2. If we take Δv and Δt smaller and smaller while still straddling a time of 1 sec.

then the approximation will get more accurate if Δv and Δt are accurate. We can take the expression for a and write it as

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \tag{1}$$

Where Δt is continually taken smaller to the limit of zero. We can actually find this limit mathematically since

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta (kt^2)}{\Delta t} = \lim_{\Delta t \to 0} \frac{[k(t + \Delta t)^2 - kt^2]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{kt^2 + 2kt\Delta t + k(\Delta t)^2 - kt^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{2kt\Delta t + k(\Delta t)^2}{\Delta t} = \lim_{\Delta t \to 0}(2kt + \Delta t) \tag{2}$$

$$= 2kt$$

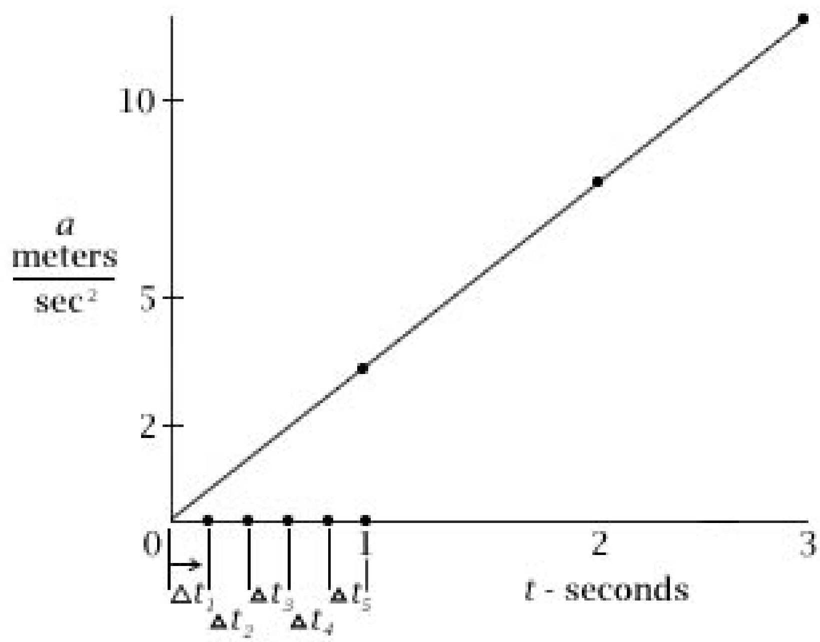
Now the acceleration at $t = 1$ sec, for the case where $k = 2$, is

$$a_{t=1} = 2(2)(1) = 4 \tag{3}$$

The expression $a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$ is the derivative of v with respect to t. It is the acceleration a and is written as dv/dt. Thus,

$$a = \frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = 2kt \qquad \text{when } v = kt^2.$$

A plot of the acceleration versus time is as shown below.



Say now we wish to find the velocity given the acceleration. Let us write the approximation of the acceleration a_{ap} as

$$a_{ap} = \frac{\Delta v}{\Delta t} \tag{4}$$

Or

$$\Delta v = a_{ap} \Delta t \tag{5}$$

If we want to find the velocity at time $t = 1$ sec let us break the time into five 0.2 sec increments. The average acceleration for the first increment is (0.8 + 0) / 2 = 0.4 and $\Delta v_1 = 0.4 \times (0.2) = 0.08$. Let us make a table for times up to 3 seconds.

Velocity Versus Time

Increment #	t	a_f	a_i	$(a_f + a_i)/2 = a$	$\Delta v = a(0.2)$	$v = \sum \Delta v$
1	0.2	0.8	0	0.4	0.08	0.08
2	0.4	1.6	0.8	1.2	0.24	0.32
3	0.6	2.4	1.6	2.0	0.40	0.72
4	0.8	3.2	2.4	2.8	0.56	1.28
5	1.0	4.0	3.2	3.6	0.72	2.00
6	1.2	4.8	4.0	4.4	0.88	2.88
7	1.4	5.6	4.8	5.2	1.04	3.92
8	1.6	6.4	5.6	6.0	1.20	5.12
9	1.8	7.2	6.4	6.8	1.36	6.48
10	2.0	8.0	7.2	7.6	1.52	8.00
11	2.2	8.8	8.0	8.4	1.68	9.68
12	2.4	9.6	8.8	9.2	1.84	11.52
13	2.6	10.4	9.6	10.0	2.00	13.52
14	2.8	11.2	10.4	10.8	2.16	15.68
15	3.0	12.0	11.2	11.6	2.32	18.00

Now we had $a = dv/dt$ and for the reverse problem we have

$$\Delta v = a\Delta t \qquad (6)$$

Knowing $a = kt$ we show in the above table how to find an approximation to v. But how do we do this in analytic form (rather than by using a numerical approach)? We can write

$$v \approx \sum \triangle v = \sum a \triangle t \tag{7}$$

The last column in the above table is the last term on the right in Eq. (7). What we do is define the integral, denoted by $\int$, of the terms to the left of the summation sign as

$$\int a dt = \lim_{\triangle t \to 0} \sum a \triangle t \tag{8}$$

How do we determine the integral, which is the inverse operation of taking the derivative? We find the integral by knowing the derivative!!! Since

$$\frac{dv}{dt} = \frac{d(kt^2)}{dt} = 2kt = a \tag{9}$$

We write this as

$$dv = d(kt^2) = 2ktdt \tag{10}$$

Now, for each term

$$\int dv = v \tag{11}$$

$$\int d(kt^2) = kt^2 \tag{12}$$

$$\int 2ktdt \quad \text{must be } kt^2 \tag{13}$$

The integral of a constant times a variable is equal to the constant times the integral of a variable. Thus

$$\int 2ktdt = 2k\int tdt \tag{14}$$

We have

$$\int tdt = \frac{t^2}{2} \tag{15}$$

We emphasize that

$$\frac{d\left(t^2/2\right)}{dt} = \frac{1}{2}\frac{dt^2}{dt} = t \tag{16}$$

Let us place these equations side-by-side

$$\frac{dt^2}{dt} = 2t \qquad\qquad \int tdt = \frac{t^2}{2} \tag{17}$$

In the above discussions we have taken the velocity as zero when the time is zero. More generally we have to add a constant c to the integral as

$$\int tdt = \frac{t^2}{2} + c \tag{18}$$

Since

$$\frac{d(t^2/2+c)}{dt} = \frac{2t}{2} + \frac{dc}{dt} = t + 0 = t \tag{19}$$

The velocity of a particle is given in terms of the displacement s by

$$\frac{ds}{dt} = v \tag{20}$$

Since $v = kt^2$ in the above example, we have

$$\begin{aligned} ds &= vdt = kt^2 dt \\ \int ds &= \int kt^2 dt \\ s &= k\frac{t^3}{3} \end{aligned} \tag{21}$$

Since

$$\frac{d(kt^3/3)}{dt} = \frac{k}{3}\frac{d(t^3)}{dt} \tag{22}$$

And

$$\begin{aligned} \frac{d(t^3)}{dt} &= \lim_{\triangle t \to 0} \frac{\triangle t^3}{\triangle t} = \lim_{\triangle t \to 0} \frac{(t+\triangle t)^3 - t^3}{\triangle t} \\ &= \lim_{\triangle t \to 0} \frac{t^3 + 3t^2 \triangle t + 3t(\triangle t)^2 + (\triangle t)^3 - t^3}{\triangle t} \\ &= \lim_{\triangle t \to 0} [3t^3 + 3t(\triangle t) + (\triangle t)^2] = 3t^2 \end{aligned} \tag{23}$$

Now

$$\frac{ds}{dt} = \frac{k}{3}\frac{d(t^3)}{dt} = \frac{k}{3} 3t^2 = kt^2 = v \tag{24}$$

Again, we know the integral of $t^2 dt$ by the derivative of t^3.

Let us summarize these results.

$$
\begin{aligned}
s &= k\frac{t^3}{3} \\
v &= \frac{ds}{dt} = \frac{d(k\,t^3/3)}{dt} = \frac{k}{3}\frac{dt^3}{dt} = kt^2 \\
a &= \frac{dv}{dt} = \frac{d(kt^2)}{dt} = 2kt
\end{aligned}
\qquad (25)
$$

The book *Differential and Integral Calculus* by Harold Maile Bacon, published by McGraw-Hill Book Company, Inc., NY, 1942, gives an extensive development of the calculus. Derivatives are developed for the most common functions and extensive integration techniques are presented. Polar coordinates are presented. The use of infinite series to solve problems is presented.

13. Advanced Calculus

Advanced calculus extends the depth and breadth of analysis in differential and integral calculus. The book *Advanced Calculus* by Ivan S. Sokolnikoff, published by McGraw-Hill Book Company, Inc., NY, 1939, contains a good treatment of these topics.

Sokolnikoff first presents a detailed discussion of the foundations of calculus. Derivatives, differentials, partial derivatives, and directional derivatives are covered. The foundation of integration is covered then integration is extended to topics such as multiple integrals, line integrals, and surface integrals. A major portion of the book is devoted to infinite series. Improper integrals and implicit functions are covered.

14. Vector and Tensor Algebra

A. Introduction

Matrices, vectors, and tensors are algebraic structures which are used to solve various physical problems. Matrices were developed primarily to solve simultaneous linear algebraic equations. Vectors representing forces, velocities, and accelerations were developed to solve mechanics problems. Tensors were developed to solve physical problems where more than one set of directions are involved. For example, in a physical situation a force could be defined by three distinct components and the resulting deflections could be defined by a different set of three components. Thus, two separate sets of directions are involved and tensors were developed to solve such problems.

B. Elementary Matrix Theory

Matrices were invented for solving simultaneous linear algebraic equations. Consider the following two equations

$$
\begin{aligned}
3x + 2y &= 8 \\
x + 4y &= 6
\end{aligned}
$$

There is a unique value for x and a unique value of y for which these two equations are satisfied.

An easy way to solve these equations is to solve the second equation for x

$$x = 6 - 4y$$

And substitute into the first equation to give

$$3(6 - 4y) + 2y = 18 - 10y = 8$$

From which

$$18 - 8 = 10y \quad \text{or} \quad y = 1$$

And

$$x = 6 - 4y = 6 - 4(1) = 2$$

Another way of solving these equations is to use matrices. This method will be seen to be more cumbersome, but for solving large numbers of simultaneous equations the matrix approach is more efficient.

Let us re-write the equations in matrix form

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

In this equation the bracketed entities are called matrices. Thus we have three matrices

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

The first matrix is a 2-row by 2-column matrix and is called a matrix having a 2 by 2 (or 2 x 2) order. Since the number of rows equals the number of columns the matrix is a square matrix. The first row has the elements 3 and 2, the second row has the elements 1 and 4. The first column has the elements 3 and 1, and the second column has the elements 2 and 4. The matrix of the unknowns $\begin{bmatrix} x \\ y \end{bmatrix}$ is called a column matrix, since it has only one column. Column matrices can be written with braces in order to save space. Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \{x \; y\}$ and $\begin{bmatrix} 8 \\ 6 \end{bmatrix} = \{8 \; 6\}$. The two column matrices $\{x \; y\}$ and $\{8 \; 6\}$ have the order 2 x 1.

Multiplication of one matrix by another requires that the matrices be conformable for multiplication. Two matrices A and B are conformable for multiplication to give the product AB if the number of columns in the pre-multiplier, matrix *A*, is equal to the number of rows in the post-multiplier, matrix *B*. Thus, the matrices

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} x \\ y \end{bmatrix}$$

are conformable for multiplication in the order *AB*, but not in the order *BA*.

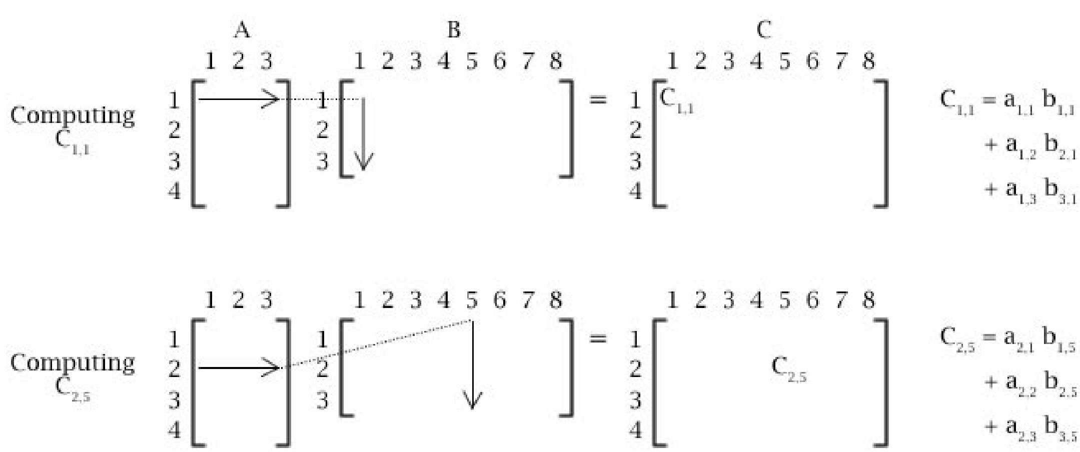
Let *C* be the product of *AB*, i.e.,

$AB = C$

Multiplication is defined so that the elements $c_{i,j}$ of *C* are $c_{1,1}$ is $a_{1,1}$ $b_{1,1}$ + $a_{1,2}$ $b_{2,1}$ and $c_{2,1}$ is $a_{2,1}$ $b_{1,1}$ + $a_{2,2}$ $b_{2,1}$ where $a_{i,j}$ and $b_{i,j}$ are elements of *A* and *B*, respectively. Thus

$$c_{1,1} = 3x + 2y$$
$$c_{2,1} = x + 4y$$

Multiplication in general is illustrated below.



Using this multiplication we can re-write the matrix equation for the two simultaneous equations as

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x+2y \\ x+4y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

Now for two matrices to be equal each element must be the same. From the above, then, we have for the first element of the two matrices on the right

$$3x + 2y = 8$$

and

$$x + 4y = 6$$

and these are the two linear algebraic equations we started with. We emphasize that

$$\begin{aligned} 3x+2y&=8 \\ x+4y&=6 \end{aligned} \quad \text{and} \quad \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

are two different ways of writing the two simultaneous equations.

In order to solve the equation metrically we define the inverse of a (square) matrix as follows

$$AA^{-1} = A^{-1}A = I$$

The symbol A^{-1} is the inverse of matrix A and I is a unit diagonal matrix. The order of A is an *mxm*, of A^{-1} is *mxm*, and of I is an *mxm*. It is clear that the number of rows and columns must be the same for all three matrices A, A^{-1}, and I.

Diagonals are defined for square matrices and the principal diagonal extends from the upper left to the lower right of the matrix. A diagonal matrix is one with non-zero elements only on the diagonal and a unit matrix is one with "1's" on each element for which $i = j$, where "i" is the row number and "j" is the column number, i.e., 1's are on the principal diagonal and zeros are everywhere else.

Examples

Diagonal matrices

$$\begin{bmatrix} 1 & & \\ & 6 & \\ & & -3 \end{bmatrix}$$

Principal diagonal matrices

$$\begin{bmatrix} 6 & & \\ & 2 & \\ & & -12 \end{bmatrix} \qquad \begin{bmatrix} 7 & \\ & -2 \end{bmatrix}$$

Unit matrices

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

With this background we now can (formally) write the solution to the matrix equation as

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

Since

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = I$$

We have

$$\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

Now if we can evaluate the inverse and pre-multiply {8, 6} by the inverse we will have a column matrix (of numbers) equal to the column matrix {x, y}. We thus will have values for x and y.

A cumbersome way of getting the inverse follows. Write the inverse as

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Now

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

Or, equating the same terms

$$3a + b = 1 \qquad 2a + 4b = 0$$
$$3c + d = 0 \qquad 2c + 4d = 1$$

Using the top two equations

$$b = 1 - 39 \quad 2a + 4(1 - 39) = 0, \qquad a = 4/10, \qquad b = -2/10$$

And the bottom two equations

$$d = -3c \qquad 2c + 4(-3c) = 1, \quad c = -1/10 \quad d = 3/10$$

Now

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}^{-1} = \frac{1}{10}\begin{bmatrix} -4 & -2 \\ -1 & 3 \end{bmatrix}$$

Post-multiplying this by the original matrix gives

$$\frac{1}{10}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 6 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, $x = 2$, $y = 1$.

One matrix can be multiplied by another (conformable for multiplication) matrix as illustrated in the above paragraphs. A matrix also can be pre-multiplied by a scalar by multiplying each element of the matrix by the scalar.

Examples

$$3\begin{bmatrix} 2 & 6 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 18 \\ 12 & 9 \end{bmatrix} \qquad a\begin{bmatrix} 2 & 6 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 2a & 6a \\ 4a & 3a \end{bmatrix}$$

Two matrices can be added (or subtracted) if they are conformable for addition. To be conformable for addition the order of the two matrices must be the same, i.e., the number of rows in each matrix must be the same and the number of columns must be the same.

Examples

$$\begin{bmatrix} 3 & 2 \\ 6 & -8 \end{bmatrix} + \begin{bmatrix} & 2 \\ 1 & \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 7 & -7 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 2 & 1 \\ 8 & 3 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 7 & 3 \\ 1 & 9 & 2 \end{bmatrix} = \begin{bmatrix} -12 & -5 & -2 \\ 7 & -6 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 9 \end{bmatrix} - \begin{bmatrix} 6 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -5 & -2 & 7 \end{bmatrix}$$

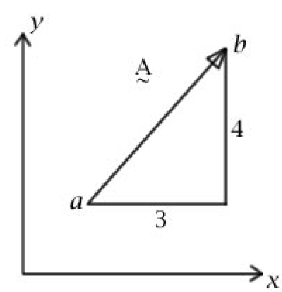
The book *Elementary Matrices* by R. A. Frazer, W. J. Duncan, and A. R. Collar, published by Cambridge Press, Cambridge, 1963, is a good introduction to matrices.

C. Elementary Vector Theory

A column matrix as well as a row matrix can represent a vector. Consider a column vector of order 2 represented by a column matrix. The vector has the properties of length and direction. Its length is simply the square root of the sum of the squares of the two elements of the order 2 matrix representing the vector. The values of the two coordinates mean that there exists an orthogonal two-dimensional coordinate system for which the vector is parallel to a plane of two of the coordinate axes. Furthermore, the value of the elements in the matrix gives the orientation and sense of the vector.

An example of a two-dimensional vector is

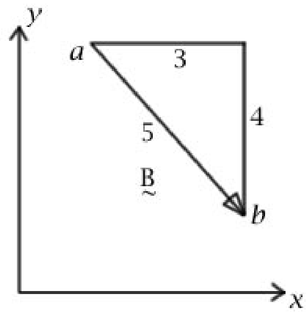
$$A = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



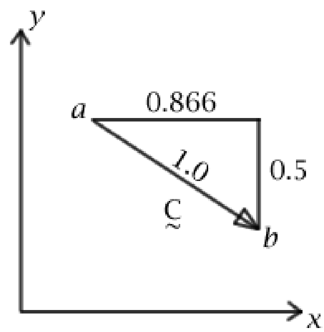
The matrix representing the vector is A which is a column matrix of order 2. The "components" of A are $A_x = 3$ and $A_y = 4$, where A is the magnitude of the vector A. The vector is shown in the figure and the notation used there for the vector uses a tilde under the letter, i.e., $\underset{\sim}{A}$. The vector starts at a and ends at b, since the x-component is +3 and the y-component is +4. The length of $\underset{\sim}{A}$ is $\sqrt{3^2 + 4^2} = 5$.

Further examples of two-dimensional vectors are

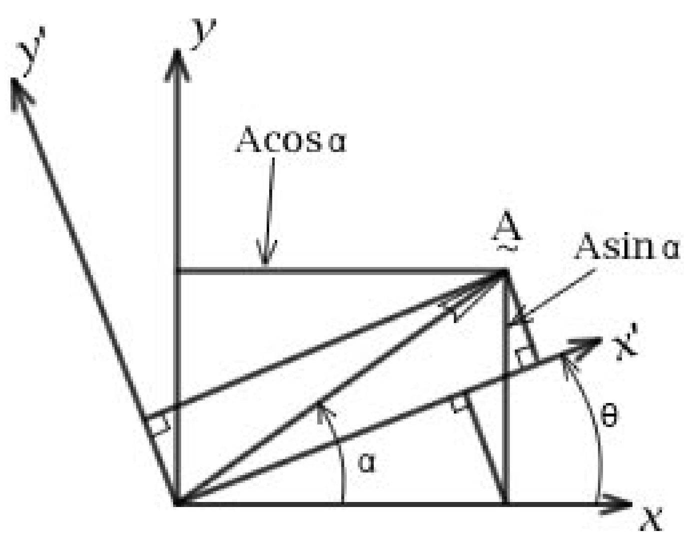
$$\underset{\sim}{B} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$



$$\underset{\sim}{C} = \begin{bmatrix} 0.866 \\ -0.5 \end{bmatrix}$$



Consider now the general case of a two-dimensional vector, see the figure below



The components of $\underset{\sim}{A}$ in the xy system are

$$A_x = A\cos\alpha$$
$$A_y = A\sin\alpha$$

The components of $\underset{\sim}{A}$ in the x'y' system are

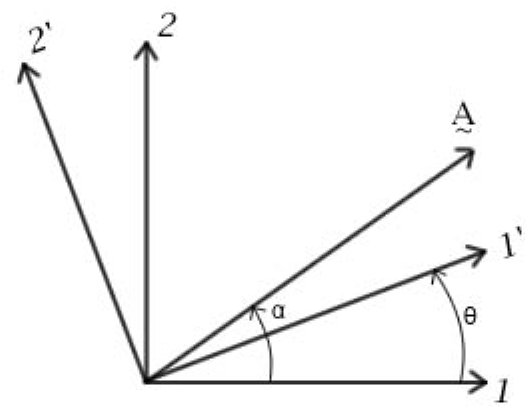
$$A_{x'} = A\cos\alpha\cos\theta + A\sin\alpha\sin\theta$$
$$A_{y'} = A\sin\alpha\cos\theta - \cos\alpha\sin\theta$$

The vector $\underset{\sim}{A}$ can now be written as

$$\underset{\sim}{A} = A\begin{bmatrix} A\cos\alpha\cos\theta + A\sin\alpha\sin\theta \\ A\sin\alpha\cos\theta - \cos\alpha\sin\theta \end{bmatrix}$$

Thus, if the components are given for any coordinate system, defined by A and α, then for any other orientation, where θ defines the orientation, the components can be determined. The immediately above matrix is the general expression for a two dimensional vector of length A.

The above expressions can be written more concisely by replacing x and y by 1 and 2, and x′ and y′ by 1′ and 2′, and by using direction cosines. We use the figure below



Define the direction cosines a_{ij} by

$a_{11'} = a_{22'} = \cos\theta$

$a_{21'} = -a_{12'} = \sin\theta$

We know

$a_{ii'} = a_{i'i}$

Further, for the 1, 2 system

$A_x = A_1 = A\cos\alpha = x_1$

$A_y = A_2 = A\sin\alpha = x_2$

And for the 1′, 2′ system

$A_{x'} = A_{1'} = A\cos(\alpha-\theta) = x_{1'}$

$A_{y'} = A_{2'} = A\sin(\alpha-\theta) = x_{2'}$

Now

$x_1' = x_1\, a_{11'} + x_2\, a_{21'}$

$x_2' = x_1\, a_{12'} + x_2\, a_{22'}$

Similarly

$x_1 = x_{1'}\, a_{1'1} + x_2\, a_{2'1}$

$$x_2 = x_{1'} a_{1'2} + x_{2'} a_{2'2}$$

These last four equations give the components of the vector in one coordinate system in terms of the components in another coordinate system.

These equations can be written even more concisely by using the "summation convention."

The last four equations can be written as

$x_{i'} = x_i \, a_{ii'}$

$x_i = x_{i'} \, a_{i'i}$

Where i' is either $1'$ or $2'$ on the left side of the first equation and i is 1 or 2 on the left side of the second equation. On the right side of each equation the repeated index (i in the first equation and "i'" in the second equation) is to be repeated (i = 1 to give the first term on the right of the first equation and i = 2 is used to give the second term). For the second equation "i'" is the repeated index. The expansions thus are

contracted	expanded
$x_{i'} = x_i \, a_{ii'}$	$x_1 = x_1 \, a_{1,1} + x_2 \, a_{2,1}$ $x_2 = x_1 \, a_{2,1} + x_2 \, a_{2,2}$
$x_i = x_{i'} \, a_{i'i}$	$x_1 = x_1 \, a_{1,1} + x_2 \, a_{2,1}$ $x_2 = x_1 \, a_{1,2} + x_2 \, a_{2,2}$

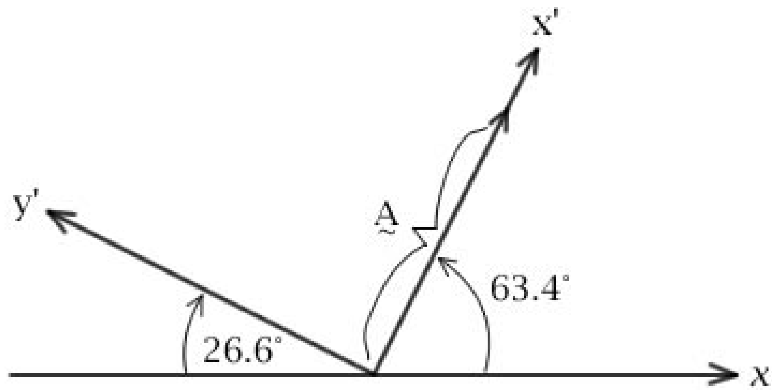
With two dimensional vectors where the two components are given, i.e., for example,

$$\underset{\sim}{A} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

it is clear that there is a coordinate system which gives a maximum element and a zero element. In this example if we let α be given as below

tan $\alpha = 8/4$, $\alpha = 63.4^\circ$

The maximizing coordinate system is shown



Further, the length of A is

$$A = \sqrt{4^2 + 8^2} = 8.94$$

Now the components of the vector in both systems are

$A_x = 4$, $A_y = 8$

$A_{x'} = 8.94$, $A_{y'} = 0$

The system of coordinates extremizing (i.e., maximizing or minimizing) the vector components are the principal coordinates (or principal axes).

Column (or row) matrices which are used to represent vectors are most often called, simply, vectors. Also, other notations are used. The matric and more common designation for a two-dimensional vector A is

Matric	(usual) Vector
$A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$	$\underset{\sim}{A} = A_x \underset{\sim}{i} + A_y \underset{\sim}{j}$

With the usual designation $\underset{\sim}{i}$ is a unit vector parallel to and in the positive sense of the x-axis and $\underset{\sim}{j}$ is a unit vector parallel to and in the positive sense of the y-axis. A_x is equal to a_1 and A_y is equal to a_2. A_x and A_y are the components of the vector parallel to the x and y axes, respectively. The magnitude of the vector $\underset{\sim}{A}$ is given by A and equals $\sqrt{{A_x}^2 + {A_y}^2}$. The ~ under *i*, *j*, and *A* implies a vector. The *A* in the matric representation means the same thing as $\underset{\sim}{A}$ in the usual vector notation. By definition

$$\underset{\sim}{A} \bullet \underset{\sim}{B} = AB\cos\alpha$$

We want to prove that

$$\underset{\sim}{A} \bullet \underset{\sim}{B} = A_x B_x + A_y B_y$$

Proof

$$\begin{aligned}
A_x B_x + A_y B_y &= A\cos(\alpha+\theta)B\cos\alpha + A\sin(\alpha+\theta)B\sin\alpha \\
&= AB[\cos(\alpha+\theta)\cos\alpha + \sin(\alpha+\theta)\sin\alpha] \\
&= AB[(\cos\alpha\cos\theta - \sin\alpha\sin\theta)\cos\alpha \\
&\quad + (\sin\alpha\cos\theta + \cos\alpha\sin\theta)\sin\alpha] \\
&= AB[\sin^2\alpha\cos\theta + \sin^2\alpha\cos\theta] \\
&= AB\cos\theta
\end{aligned}$$

The vector component approach for calculating the dot product simplifies the procedure somewhat for two-dimensional problems. For three dimensions the calculations are greatly simplified.

D. Elementary Tensor Theory

Let us now define tensors and distinguish them from vectors and matrices. First let us precisely define a matrix.

A matrix is a rectangular array of numbers. In general there are "m" rows and "n" columns. The number of rows can be any value as low as "1" and can vary up to any positive integer. Likewise "n" can be any value from "1" to any other positive integer. Each element can be any number. There is not necessarily any relation between any of the elements.

A vector is either a column matrix (or one column of a matrix) or it can be a row matrix (or one row of a matrix). In general there is not necessarily any relation between any of the elements. However, the term "vector" most commonly refers to its use to represent the geometric concept of a line segment directed from starting at one point and ending at another point. In this case the vector is a two or three element matrix. However, while the (geometric) vector is a matrix it has a special invariant property. The elements are related to a coordinate system, and the invariant property is that the sum of the squares of the elements is invariant with coordinate rotation.

A tensor of rank 2 relates one vector to a second vector whose magnitude and direction may differ from the first vector. Each vector has the characterizing property that the sum of the squares of the terms is independent of the coordinate system. The tensor consists of three square matrices all of the same order multiplied together. For physical problems elements of the central matrix give the simplified basic relations making up the relationships between the two vectors. The magnitudes of the elements in the central matrix depend upon the orientation of the coordinate axis system. The reference coordinate system is the “principal” axis system which is the system which gives zero values except on the diagonal, i.e., the central matrix referred to the principal axes is a diagonal matrix. For other orientations the matrix can be non-diagonal. The pre-multiplier and post-multiplier matrices are diagonal matrices whose elements are unit vectors – each parallel to one of the coordinate axes. We use the notation $\underset{\approx}{\psi}$ for a tensor. Thus, a three-dimensional second rank tensor appears as

$$\underset{\approx}{\psi} = \begin{bmatrix} \underset{\sim}{i} & & \\ & \underset{\sim}{j} & \\ & & \underset{\sim}{k} \end{bmatrix} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} \underset{\sim}{i} & & \\ & \underset{\sim}{j} & \\ & & \underset{\sim}{k} \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\sim}{i} & & \\ & \underset{\sim}{j} & \\ & & \underset{\sim}{k} \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} \underset{\sim}{i} & & \\ & \underset{\sim}{j} & \\ & & \underset{\sim}{k} \end{bmatrix}$$

Letting $\underset{\sim}{B}$ be the “operated on” vector and $\underset{\sim}{C}$ be the result of the operation we have

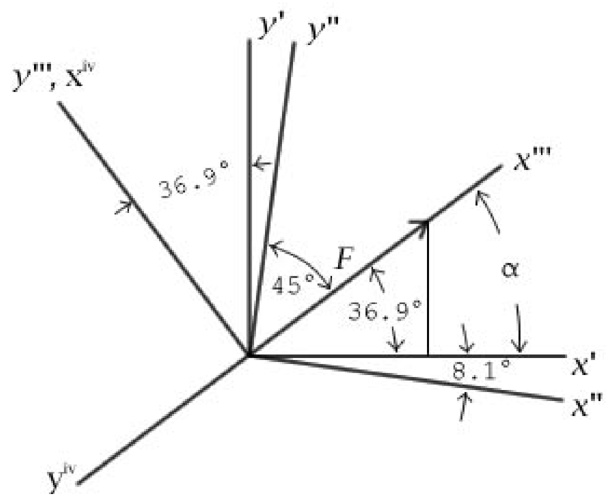
$$\underset{\sim}{C} = \underset{\approx}{\psi}\underset{\sim}{B}$$

Vectors $\underset{\sim}{B}$ and $\underset{\sim}{C}$ will be referred to the coordinate system defined by the limit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$, and $\underset{\sim}{k}$.

In dealing with tensors we often need to transform a vector from one set of Cartesian coordinates to a system which has a different orientation. In the following section we derive the equations for transforming coordinates for vectors.

a. Transformation of Coordinates for a Two Dimensional Vector

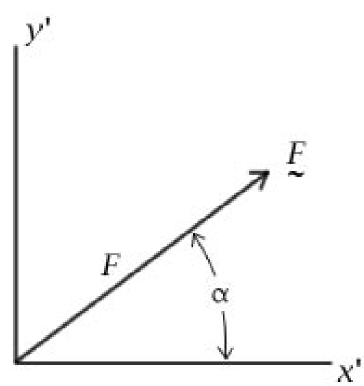
The following figure shows a two-dimensional vector $\underset{\sim}{F}$



and four two-dimensional coordinate systems. Obviously there are an unlimited number of coordinate systems for representing $\underset{\sim}{F}$ since α can be any real number from 0 to 360˚.

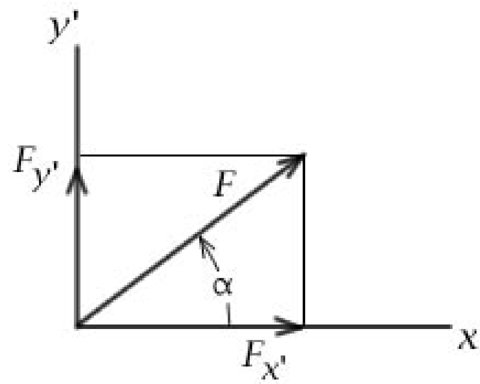
The two elements in the matrices are the "components" of the vector for the particular coordinate system selected. For the *x*'*y*' system the two components are

$$F_{x'} = F\cos\alpha$$
$$F_{y'} = F\sin\alpha$$



where *F* is the magnitude of the vector.

The components of F thus are shown below.



For the case where $F = 5$ for the following values of α we have

$$F = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad F = \begin{bmatrix} \sqrt{2.5} \\ \sqrt{2.5} \end{bmatrix}, \quad F = \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \text{ and } F = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$\alpha = 36.9°$ $\quad\alpha = 45°$ $\quad\alpha = 90°$ $\quad\alpha = 0°$

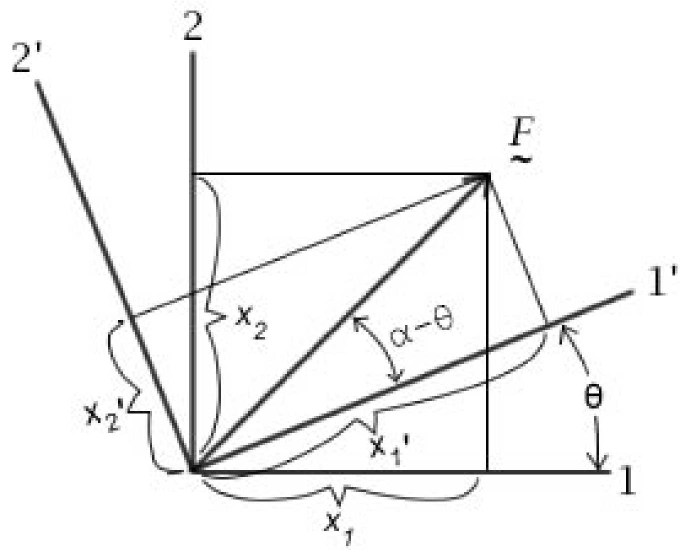
All of these are "equivalent" vectors.

The expression

$$\underset{\sim}{F} = \begin{bmatrix} A\cos\alpha \\ A\sin\alpha \end{bmatrix}$$

denotes the whole system of vectors where $0 \le \alpha \le 360^{\circ}$.

If $\underset{\sim}{F}$ is given for one set of coordinates then $\underset{\sim}{F}$ can be determined for another set (where the coordinate systems are coplanar and have a common origin) using "transformation" equations. The following figure shows $\underset{\sim}{F}$ and two coordinate systems.



Letting F be the magnitude of $\underset{\sim}{F}$ we have

$$x_1 = F \cos \alpha \qquad x_2 = F \sin \alpha$$

and

$$x_{1'} = F \cos (\alpha - \theta) \qquad x_{2'} = F \sin (\alpha - \theta)$$

Now letting

$$a_{11'} = a_{22'} = \cos \theta , \quad a_{21'} = -a_{12'} = \sin \theta$$

we have

$$x_{1'} = x_1 a_{11'} + x_2 a_{21'} = x_1 \cos \theta + x_2 \sin \theta$$

or

$$x_{1'} = F (\cos \alpha \cos \theta + \sin \alpha \sin \theta) = F \cos (\alpha - \theta)$$

*

Similarly

$$x_{2'} = x_1 a_{12'} + x_2 a_{22'} = -x_1 \sin \theta + x_2 \cos \theta$$

or

$$x_{2'} = F (\sin \alpha \cos \theta - \cos \alpha \sin \theta) = F \sin (\alpha - \theta)$$

In the same way for x_1 and x_2 we have

$$\left.\begin{aligned} x_1 &= x_{1'}\, a_{1'1} + x_{2'}\, a_{2'1} = x_{1'} \cos\theta - x_{2'} \sin\theta \\ x_2 &= x_{1'}\, a_{1'2} + x_{2'}\, a_{2'2} = x_{1'} \sin\theta + x_{2'} \cos\theta \end{aligned}\right\} \; **$$

A two-dimensional vector is a system whose components in one coordinate system transform to another common origin coplanar coordinate system as determined by the equations * or ** above.

A summation convention is introduced for use in abbreviating the description of vectors. We write

$$x_{i'} = x_i\, a_{ii} = x_1\, a_{1i'} + x_2\, a_{2i'}$$

where i' stands for either $1'$ or $2'$. If $i = 1'$, the summation convention applied to the repeated index i transforms the above equation into

$$x_{1'} = x_1\, a_{11'} + x_2\, a_{21'}$$

and for $i = 2'$ we obtain

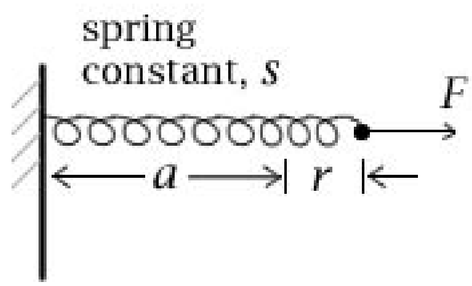
$$x_{2'} = x_1\, a_{12'} + x_2\, a_{22'}$$

We usually write x_i in the 1, 2 system and $x_{i'}$ in the $1'$, $2'$ system.

b. Use of Tensors for Solving a Mechanical Problem

We consider a simple mechanical problem which is easy to solve without tensors. We then solve the same problem using tensors. This tensor solution illustrates how tensors are used to calculate a deflection which has a direction different from the direction of the force. While the use of tensors with this simple problem is more cumbersome than the usual technique, there are problems where tensors are almost indispensable. This simple problem develops the tensor technique.

We begin with a “coil” spring fixed at one end and loaded axially at the other end, see the figure. In this figure

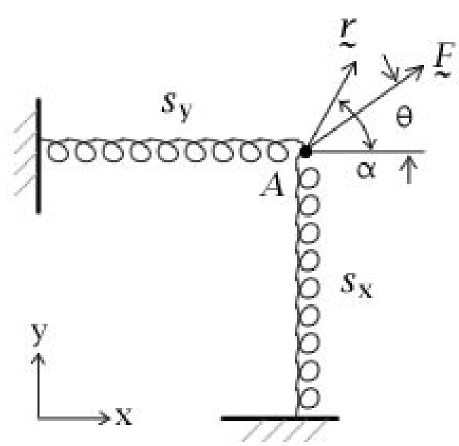


the unloaded length is “a” and the loaded length is $a + r$. The relation between the force and deflection is

$$F = s\,r$$

where s is the stiffness, or the “*spring constant.*”

Consider now the two-dimensional case shown in the next figure.



The component in the x-direction of the force is

$$F_x = F\cos\theta$$

And in the y-direction it is

$F_y = F \sin \theta$

The deflections in the two directions are

$r_x = F_x / s_x = F \cos \theta / s_x$

$r_y = F_y / s_y = F \sin \theta / s_y$

Now, the total deflection is

$$r = \sqrt{r_x^{\,2} + r_y^{\,2}} = F \sqrt{\frac{\cos^2 \theta}{s_x^{\,2}} + \frac{\sin^2 \theta}{s_y^{\,2}}}$$

The direction of the deflection is given by

$$\tan \alpha = \frac{s_x}{s_y} \tan \theta$$

Let us now apply tensors to this problem. The deflection can be represented by a vector $\underset{\sim}{r}$ whose "tail" is at the undeflected position of *A* in the figure and whose "head" is at the deflected position of *A*. Now

$$\underset{\sim}{r} = \underset{\sim}{i}\, r_x + \underset{\sim}{j}\, r_y$$

Using the vector notation for the force we have

$$\underset{\sim}{F} = \underset{\sim}{i}\, F_x + \underset{\sim}{j}\, F_y$$

we can write F_x and F_y as

$$F_x = s_x x \qquad F_y = s_y y$$

Now

$$\begin{aligned}\underset{\sim}{F} &= \underset{\sim}{i}F_x + \underset{\sim}{j}F_y \\ &= \underset{\sim}{i}\left(s_x r_x\right) + \underset{\sim}{j}\left(s_y r_y\right) \\ &= \underset{\sim}{i}\left[s_x\left(\underset{\sim}{i}\cdot\underset{\sim}{r}\right)\right] + \underset{\sim}{j}\left[s_y\left(\underset{\sim}{j}\cdot\underset{\sim}{r}\right)\right] \\ &= s_x\left[\underset{\sim}{i}\left(\underset{\sim}{i}\cdot\underset{\sim}{r}\right)\right] + s_y\left[\underset{\sim}{j}\left(\underset{\sim}{j}\cdot\underset{\sim}{r}\right)\right] \\ &= \left[s_x\underset{\sim}{i}\underset{\sim}{i} + s_y\,\underset{\sim}{j}\,\underset{\sim}{j}\right]\cdot\underset{\sim}{r}\end{aligned}$$

Until this last step all of the operations are straightforward operations previously defined.

Let us look more carefully at the equation from the next to the last step.

$$\underset{\sim}{F} = k_x\left[\underset{\sim}{i}\left(\underset{\sim}{i}\cdot\underset{\sim}{r}\right)\right] + k_y\left[\underset{\sim}{j}\left(\underset{\sim}{j}\cdot\underset{\sim}{r}\right)\right]$$

The terms in the parentheses are the (scalar) components of the deflections in the x- and y-directions. Pre-multiplying these by $\underset{\sim}{i}$ and $\underset{\sim}{j}$, respectively then multiplying by the spring constants gives the vector components of the force. In the next step all that is done is to factor out the vector $\underset{\sim}{r}$ so that we have

$$\underset{\sim}{F} = \left[k_x\underset{\sim}{i}\underset{\sim}{i} + k_y\,\underset{\sim}{j}\,\underset{\sim}{j}\right]\cdot\underset{\sim}{r} \qquad (***)$$

where we see the previously undefined operations $\underset{\sim}{i}\underset{\sim}{i}$ and $\underset{\sim}{j}\underset{\sim}{j}$. The steps in using the bracketed terms are clear, the second vector in each term is dotted into the vector $\underset{\sim}{r}$ simply giving the force vector.

If we compare the two expressions

$$\underset{\sim}{F} = \left[s_x \underset{\sim}{i}\underset{\sim}{i} + s_y \underset{\sim}{j}\underset{\sim}{j} \right] \cdot \underset{\sim}{r}$$

and

$$\underset{\sim}{F} = \underset{\sim}{i}\left[s_x \left(\underset{\sim}{i} \cdot \underset{\sim}{r} \right) \right] + \underset{\sim}{j}\left[s_y \left(\underset{\sim}{j} \cdot \underset{\sim}{r} \right) \right]$$

Then expand the second expression giving

$$\underset{\sim}{F} = \underset{\sim}{i}\left[s_x \left(\underset{\sim}{i} \cdot \left\{ \underset{\sim}{i}\, r_x + \underset{\sim}{j}\, r_y \right\} \right) \right] + \underset{\sim}{j}\left[s_y \left(\underset{\sim}{j} \cdot \left\{ \underset{\sim}{i}\, r_x + \underset{\sim}{j}\, r_y \right\} \right) \right]$$

$$= \underset{\sim}{i}\, s_x \left[\underset{\sim}{i} \cdot \underset{\sim}{i}\, r_x \right] + \underset{\sim}{j}\left[s_y\, \underset{\sim}{j} \cdot \underset{\sim}{j}\, r_y \right] = s_x r_x\, \underset{\sim}{i} + s_y r_y\, \underset{\sim}{j}$$

Now expanding the first expression

$$\underset{\sim}{F} = s_x\, \underset{\sim}{i}\,\underset{\sim}{i} \cdot \left(\underset{\sim}{i}\, r_x + \underset{\sim}{j}\, r_y \right) + s_y\, \underset{\sim}{j}\,\underset{\sim}{j} \cdot \left(\underset{\sim}{i}\, r_x + \underset{\sim}{j}\, r_y \right)$$

Comparing these last two expressions we see that

$$s_x r_x \underset{\sim}{i} = s_x \left[\underset{\sim}{i} \cdot \left(\underset{\sim}{i} r_x + \underset{\sim}{j} r_y \right) \right] \underset{\sim}{i}$$

and

$$s_y r_y \underset{\sim}{j} = s_y \left[\underset{\sim}{j} \cdot \left(\underset{\sim}{i} r_x + \underset{\sim}{j} r_y \right) \right] \underset{\sim}{j}$$

or

$$r_x = \underset{\sim}{i} \cdot \underset{\sim}{i} r_x + \underset{\sim}{i} \cdot \underset{\sim}{j} r_y = (1) r_x + (0) r_y = r_x$$

and

$$r_y = \underset{\sim}{j} \cdot \underset{\sim}{i} r_x + \underset{\sim}{j} \cdot \underset{\sim}{j} r_y = (0) r_x + (1) r_y = r_y$$

What this means is that if we return to Equation (***) to evaluate the terms $\underset{\sim}{i}\underset{\sim}{i}$ and $\underset{\sim}{j}\,\underset{\sim}{j}$ we dot the second vector into $\underset{\sim}{r}$ giving a scalar times the first $\underset{\sim}{i}$ for the first term and a scalar times the first $\underset{\sim}{j}$ for the second term – the sum of which, of course, is a vector.

The term in the brackets of (***) is a tensor of the second rank. It is called the second rank since there are two sets of directions involved – the direction of the force components and the direction of the deflection components. We use the notation

$$\underset{\approx}{\psi} = s_x \underset{\sim}{i}\underset{\sim}{i} + s_y \underset{\sim}{j}\underset{\sim}{j}$$

Where $\underset{\approx}{\psi}$ is the tensor of the second rank. When $\underset{\approx}{\psi}$ is post-dot multiplied by the vector $\underset{\sim}{r}$ the result is a vector. Thus, we have

$$\underset{\approx}{\psi} \cdot \underset{\sim}{r} = \underset{\sim}{F}$$

As a final step let us write the resulting equation in vector form.

$$\underset{\sim}{F} = \underset{\approx}{\psi} \cdot \underset{\sim}{r} = (s_x \underset{\sim}{i}\underset{\sim}{i} + s_y \underset{\sim}{j}\underset{\sim}{j}) \cdot (r_x \underset{\sim}{i} + r_y \underset{\sim}{j}) = s_x \underset{\sim}{i} r_x + s_y \underset{\sim}{j} r_y \qquad (****)$$

$$= \underset{\sim}{i} s_x r_x + \underset{\sim}{j} s_y r_y$$

A tensor of rank 2 post-dot multiplied by a vector gives a vector which is a tensor of rank 1. Also, recall that a vector post-dot multiplied by a vector produces a scalar which is a tensor of rank "0."

Let us re-write the expression relating the force vector to the displacement vector. We have

$$\underset{\sim}{F} = \underset{\approx}{\psi} \cdot \underset{\sim}{r} = [s_x \underset{\sim}{i}\underset{\sim}{i} + s_y \underset{\sim}{j}\underset{\sim}{j}] \cdot \underset{\sim}{r}$$

$$= \begin{bmatrix} \underset{\sim}{i} & \\ & \underset{\sim}{j} \end{bmatrix} \begin{bmatrix} s_x & \\ & s_y \end{bmatrix} \begin{bmatrix} \underset{\sim}{i} & \\ & \underset{\sim}{j} \end{bmatrix} \cdot \underset{\sim}{r}$$

For review, let us expand these matrices

$$\begin{bmatrix} \underset{\sim}{i} & \\ & \underset{\sim}{j} \end{bmatrix} \begin{bmatrix} s_x & \\ & s_y \end{bmatrix} \begin{bmatrix} \underset{\sim}{i} & \\ & \underset{\sim}{j} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{i} s_x & \\ & \underset{\sim}{j} s_y \end{bmatrix} \begin{bmatrix} \underset{\sim}{i} & \\ & \underset{\sim}{j} \end{bmatrix} = \begin{bmatrix} s_x \underset{\sim}{i}\underset{\sim}{i} & \\ & s_y \underset{\sim}{j}\underset{\sim}{j} \end{bmatrix}$$

Thus

$$\underset{\sim}{F} = \underset{\approx}{\psi} \cdot \underset{\sim}{r} = \begin{bmatrix} s_x \underset{\sim}{i}\underset{\sim}{i} & \\ & s_y \underset{\sim}{j}\underset{\sim}{j} \end{bmatrix} \cdot \underset{\sim}{r} = \begin{bmatrix} \underset{\sim}{i} & \\ & \underset{\sim}{j} \end{bmatrix} \begin{bmatrix} s_x & \\ & s_y \end{bmatrix} \begin{bmatrix} \underset{\sim}{i} & \\ & \underset{\sim}{j} \end{bmatrix} \cdot \underset{\sim}{r}$$

The matrix

$$\begin{bmatrix} s_x & \\ & s_y \end{bmatrix}$$

is the "stiffness" matrix. The larger s_x and s_y are, the stiffer the spring system. The element s_x is the amount of force applied in the x-direction to produce a unit deflection in the x-direction. We could have used the notation s_{xx}, instead of s_x, where the first x refers to the direction of the force and the second x refers to the direction of the displacement. Similarly we could use s_{yy} for s_y.

Let us now summarize the above. A vector is either a column matrix (or one column of a matrix) or it can be a row matrix (or one row of a matrix). In general there is not necessarily any relation between the elements. However, the term *vector* most commonly refers to its use to represent the geometric concept of a line segment directed from a starting point and ending at another point.

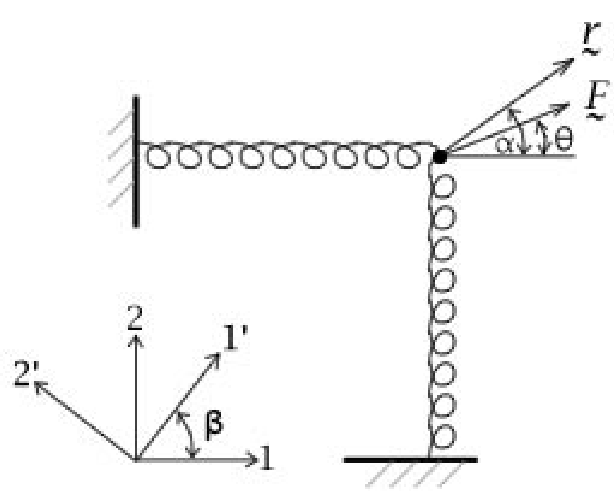
A (geometric) vector normally is specified by using a rectangular (Cartesian) coordinate system. If the vector is parallel to the plane formed by the two axes then the vector will have either one or two non-zero elements. The sum of the squares of these two elements will be invariant for all two dimensional coordinate systems whose planes are parallel to the initial plane. If a two-dimensional coordinate system is used for which the vector is not parallel to the plane formed by its axes, then a third orthogonal axis must be added in order to specify the vector. The vector representation for this three-dimensional system, in general, will have three components. The sum of the squares of the three components will be the same independent of the angular orientation of any coordinate system selected.

A geometric vector represented by coordinate system for which one plane is always parallel to the vector is called a two dimensional vector, and its non-zero elements will be one or two, depending upon the orientation of the system. A geometric vector represented by coordinate systems for which the vector can be non-parallel to none of the three planes defined by the coordinate axes is called a three-dimensional vector.

Some mechanical systems have forces and deflections both of which are parallel to a particular plane. The vector representing the force and the vector representing the deflection thus are two-dimensional vectors.

c. The Two-Spring Problem Referred to a Rotated Coordinate System

We now consider the coordinate system shown below.



Refer back to the transformation equations. We write $\underset{\sim}{r}$ in the 1, 2 coordinate system and the $1'$, $2'$ system.

$$r_{1'} = r_1 \, a_{1,1'} + r_2 \, a_{2,1'}$$

$$r_{2'} = r_1 \, a_{1,2'} + r_2 \, a_{2,2'}$$

and

$$\underset{\sim}{r}' = \underset{\sim}{i}' r_{1'} + \underset{\sim}{j}' r_{2'} = \underset{\sim}{i}'\left(r_1 a_{1,1'} + r_2 a_{2,1'}\right) + \underset{\sim}{j}'\left(r_1 a_{1,2'} + r_2 a_{2,2'}\right)$$

Where

$$a_{1,1'} = a_{2,2'} = \cos\beta \ , \qquad a_{2,1'} = -a_{1,2'} \sin\beta$$

and we prime the $\underset{\sim}{r}$ to show that it is referred to the $'$ system. We can write $\underset{\sim}{r}'$ in matric form as

$$\underset{\sim}{r}' = \left[\underset{\sim}{i}', \underset{\sim}{j}'\right]\left[r_1, r_2\right]\begin{bmatrix} a_{1,1'}, a_{1,2'} \\ a_{2,1'}, a_{2,2'} \end{bmatrix}$$

Similarly we can express $\underset{\sim}{F}$ in the transformed coordinate system as

$$\underset{\sim}{F}' = \left[\underset{\sim}{i}', \underset{\sim}{j}'\right]\left(\left[F_1, F_2\right]\begin{bmatrix} a_{1,1'}, a_{1,2'} \\ a_{2,1'}, a_{2,2'} \end{bmatrix}\right)$$

Let us summarize:

> i' and j' are unit vectors orthogonal with the new coordinate system.

r_1 and r_2 as well as F_1 and F_2 are the components of the vectors in the original coordinate system.

The matrix $A = \begin{bmatrix} a_{1,1'}, a_{1,2'} \\ a_{2,1'}, a_{2,2'} \end{bmatrix}$ gives the direction cosines.

Thus, $\underset{\sim}{r}'$ and $\underset{\sim}{F}'$ are the vectors $\underset{\sim}{r}$ and $\underset{\sim}{F}$ expressed in the new coordinate system.

The book *Vector and Tensor Analysis* by Entiquio Young, published by Marcel Dekker, Inc. NY, 1978 extends the above topics in great depth. In addition, numerous problems are given so that the reader can develop great proficiency in the subjects.

15. Ordinary Differential Equations

Ordinary differential equations are equations in which derivatives of functions occur where the function depends only upon one variable. The derivatives can be first order, or of higher order. Strictly speaking, the equations $dy/dt = t$ or $d^2y/dt^2 = t^3$ are differential equations. However, when the variables can be separated easily to be put in forms which can be integrated by the methods of calculus then the name differential equation is not used. In the above instances we have

$$\frac{dy}{dt} = t \;, \qquad dy = tdt \;, \qquad \int dy = \int tdt \;, \qquad y = \frac{t^2}{2} + c$$

$$\frac{d^2y}{dt^2} = t^3 \;, \qquad d\left(\frac{dy}{dt}\right) = t^3 dt \;, \qquad \frac{dy}{dt} = \frac{t^4}{4} + c_1 ,$$

$$dy = \frac{t^4}{4} dt + c_1 dt \;, \qquad \int dy = \int \frac{t^4}{4} dt + \int c_1 dt \;,$$

$$y = \frac{t^5}{20} + c_1 t + c_2$$

The derivatives in differential equations can be raised to different powers – which determines their degree. The order of a differential equation is the highest derivative in the equation. If the highest derivative is d^3y/dt^3 then the differential equation is of third order.

The object of finding a solution to a differential equation is to obtain an algebraic expression which does not contain any derivatives and which algebraic equation is the most general equation which satisfies the original differential equation.

The solutions of ordinary differential equations are obtained using the same approach used for integral calculus. Solutions that have previously been obtained are organized in various manners so that a particular differential equation can be connected to the solution. For some differential equations a new analytic function, such as an elliptic integral, must be defined to obtain a solution.

Elementary Differential Equations by Lyman M. Kells, published by McGraw-Hill, NY 1935, gives a “first start” of techniques for solving ordinary equations.

16. Complex Variables

Complex numbers were first introduced into mathematics by the requirement to solve for the values of the variable in a polynomial which made the polynomial have a value of zero. In the subsequent development of advanced analysis complex variables became indispensable. Complex variables pervade practically all of advanced analysis. The book *Complex Variables and Applications* (second edition) by Ruel V. Churchill, published by McGraw-Hill, NY 1960, presents a good introduction to the subject.

17. Fourier Series

An elastic body can be subjected to the cyclic action of a force versus time of any shape. It was discovered by Jean Baptiste Joseph Fourier (1768 – 1830) that cyclical forcing functions of practically any variation with time could be represented with sufficient accuracy by a superposition of a number of sine (or cosine) functions of time. The resulting functions resulting from superpositions of these sine and cosine functions are called Fourier series.

Fourier Series and Boundary Value Problems by Ruel V. Churchill, published by McGraw-Hill, NY 1941, is an introduction to this theory. The following paragraphs are excerpted from the preface of this book.

This is an introductory treatment of Fourier series and their application to the solution of boundary value problems in the partial differential equations of physics and engineering. It is designed for students who have had an introductory course in ordinary differential equations and one semester of advanced calculus, or an equivalent preparation. The concepts from the field of physics which are involved here are kept on an elementary level. They are explained in the early part of the book, so that no previous preparation in this direction need be assumed.

The first objective of this book is to introduce the reader to the concept of orthogonal sets of functions and to the basic ideas of the use of such functions in representing arbitrary functions. The most prominent special case, that of representing an arbitrary function by its Fourier series, is given special attention. The Fourier integral representation and the representation of functions by series of Bessel functions and Legendre polynomials are also treated individually, but somewhat less fully. The material covered is intended to prepare the reader for the usual applications arising in the physical sciences and to furnish a sound background for those who wish to pursue the subject further.

The second objective is a thorough acquaintance with the classical process of solving boundary value problems in partial differential equations, with the aid of those expansions in series of orthogonal functions. The boundary value problems treated here consist of a variety of problems in heat conduction, vibration, and potential. Emphasis is placed on the formal method of obtaining the solutions of such problems. But attention is also given to the matters of fully establishing the results as solutions and of investigating their uniqueness, for the process cannot be properly presented without some consideration of these matters.

The book is intended to be both elementary and mathematically sound. It has been the author's experience that careful attention to the mathematical development, in contrast to more formal procedures, contributes much to the student's interest as well as to his understanding of the subject, whether he is a student of pure or of applied mathematics. The few theorems that are stated here without proofs appear at the end of the discussion of the topics concerned, so they do not reflect upon the completeness of the earlier part of the development.

Illustrative examples are given whenever new processes are involved.

The problems form an essential part of such a book. A rather generous supply and wide variety will be found here. Answers are given to all but a few of the problems.

The chapters on Bessel functions and Legendre polynomials (Chaps. VIII and IX) are independent of each other, so that they can be taken up in either order. The continuity of the subject matter will not be interrupted by omitting the chapter on the uniqueness of solutions of boundary value problems (Chap. VII) or by omitting certain parts of other chapters.

18. Operational Mathematics

In operational mathematics an "operator" is defined which when operating upon a differential equation of the first order transforms the equation to an algebraic equation. The primary operator is the Laplace transform.

A significant problem which is solved using operational mathematics is the response of an elastic body to a force impulse of short duration.

The book *Modern Operational Mathematics in Engineering* by Ruel V. Churchill, published by McGraw-Hill, NY 1944 is a good practical introduction to the subject. The following paragraphs are excerpted from the preface of this book.

This is a textbook on applications of the Laplace transformation. The applications are chiefly to problems in engineering and physics that involve differential equations, with emphasis on boundary value problems in partial differential equations.

In partial differential equations the method of using the Laplace transformation, which is the operational method, and the more classical one of using Fourier series supplement one another. Thus this book is a companion volume to the author's earlier book entitled "Fourier Series and Boundary Value Problems."

No previous preparation in the subject of partial differential equations is required of the student; in fact very little previous experience with ordinary differential equations is actually needed here. A year of college physics would seem to furnish a sufficient background for the physical and engineering problems treated here, since these problems are kept on a fairly elementary level.

The first four chapters of this book are somewhat more elementary than the material in the earlier volume. These chapters include a treatment of problems in ordinary as well as partial differential equations. They have served as a textbook for a short basic course in operational mathematics given to juniors and seniors at the University of Michigan and to research engineers in Detroit. Chapter X, which represents the operational properties of finite Fourier transformations, can well be included in such a course.

The remaining chapters are more advanced. They make use of certain material, described in Chapter V, from the theory of functions of a complex variable. For several years the material in these chapters has been included in a graduate course on methods in partial differential equations.

Important results are stated as theorems. Thus the reader who is primarily interested in engineering or physics rather than in applied mathematics should find it possible to skip over details in mathematical derivations. An attempt has been made to keep the mathematical analysis on a level that is as elementary as possible without impairing the usefulness of the results. Such a program tends to make some developments more tedious, however, than they need be if more advanced concepts in function theory had been used.

19. Numerical Analysis

What is numerical analysis? A polynomial with one variable of the first, second, third, and fourth degree (i.e., 1^{st}, 2^{nd}, 3^{rd}, or 4^{th} power) can be solved in closed form where all the factors multiplying the unknown variable are given as algebraic numbers. Polynomials of the 5^{th} power, and higher, generally must be solved numerically. However, this is not generally thought of as numerical analysis.

A system of one, two, and three linear equations in one, two, or three variables, respectively, can be solved in terms of the coefficients of the variables. Solutions for linear systems much beyond that number of unknowns become unwieldy and, usually, are solved only where the coefficients of the unknowns have numerical values and the solutions are obtained numerically. In the 1950's in the United States aircraft manufacturers were solving simultaneous linear equations with 100 variables. In the 1960's the number of variables increased to 1000. Solutions with many unknowns were made possible by the development of large-scale computers.

Numerical solution techniques, however, most often are associated with the solution of ordinary and partial differential equations. Rocket-powered missile trajectories were routinely solved numerically in the late 1940's and early 1950's with simple calculators. Also structural vibrational responses to rocket start-up forces were solved using simple calculators to solve the resulting ordinary differential equations. In the late 1940's and early 1950's, simplified solutions of structural and heat transfer problems, for example, were solved numerically. These problems involved partial differential equations and usually had only a few finite elements – and thus only a few equations to be solved.

In the mid 1950's as high speed (powerful) computers were developed some scientists felt there was no practical problem that could not be solved numerically. However there are limitations that were with us then, are still present today (60 years later), and may be still present 60 years later!! Practical engineering problems involve continua which are made up of many atoms. Even with the action of the atoms known it is difficult to determine properties of an elementary element made up of billions of billions of atoms. Such properties of continua are the equations-of-state. The problem is even more difficult at the sub-atomic level and even more so at the most basic level of the single type elementary particle making everything in the universe (assuming such a particle exists).

The book *Numerical Analysis* 8th edition, (ISBN 0-534-39200-8) by Richard L. Burden and J. Douglas Faires, published by Thomson/Brooks/Cole, Belmont, CA 2005, presents current numerical techniques for solving mathematical problems. *Numerical Calculus* by William Edmund Milne, published by Princeton University Press, Princeton, New Jersey 1949, is an early easily readable introduction to numerical analysis.

20. Vector Mechanics for Engineers

All the previous referenced books have dealt with topics underlying the development of physical science. *The Foundations of Language* lays out the postulates of physics as well as the postulates from which language, logic, and mathematics are derived. The next referenced books develop the mathematics required for deriving the physical theorems from the physical postulates in *Foundations of Language*. Using these results including the postulated particle and its characteristics we make the assumption that the basic particles of physics can form stable assemblages and thus form the physical things we observe in the universe. The book *Vector Mechanics for Engineers* 6^{th} edition, by Ferdinand P. Beer and E. Russell Johnston, Jr. (ISBN 0072446110), published by McGraw-Hill, NY 2001, develops techniques for analyzing particles and rigid bodies at rest and when moving. The techniques are developed in great detail and there are many problems which can be solved to solidify the content for the diligent researcher.

21. Kinetic Theory of Gases, Statistical Mechanics, and Thermodynamics

In theoretical analyses of particle mechanics the individual particles are considered or assemblies of particles are analyzed without the particles having any relation specified for their interaction among themselves. In kinetic theory and thermodynamics we use "equations of state." We consider a volume containing many particles and we call the assemblage of particles a "gas." The gas, in kinetic theory, is characterized by pressure-volume-temperature (i.e., internal energy) relations. Thus instead of dealing with many particles the analyses deal with a (very) few parameters.

The statistical mechanics approach differs from the kinetic theory approach to the analyses of gases. In kinetic theory the actual mechanics of interactions between two particles of the gas are used to evolve the equation of state. In statistical mechanics a phase space is defined which consists of a volume of "six-dimensional space" which lies between an increment of the x-dimension, an increment of the y-dimension, an increment of the z-dimension, an increment of the x-component of particle velocity, an increment of the y-component of particle velocity, and finally an increment of the z-component of particle velocity. A function "n" is defined as the expected (or probable) number of particles then in that element of phase space. An elementary statistical mechanics assumption then is made that the expected number of particles in a given increment of phase space depends only upon the local overall properties of the gas – typically the pressure, temperature, and density. Subject to these conditions the expected number of particles is equally-likely.

In thermodynamics, we have kinetic theory but in addition we have heat flowing in and out of the gas.

An Introduction to Thermodynamics, The Kinetic Theory of Gases, and Statistical Mechanics 2nd edition, by Francis Weston Sears, published by Addison-Wesley, Reading, Massachusetts 1963, is an excellent elementary introduction to these subjects. Chapters 11, 12, and 13 cover kinetic theory; Chapters 14 and 15 cover statistical mechanics; and Chapters 1-10 present the elementary concepts of thermodynamics.

22. Advanced Kinetic Theory of Gases

Introduction to Physical Gas Dynamics by Walter G. Vincenti and Charles H. Kruger, Jr. published by John Wiley and Sons, Inc. New York, 1965 contains a good exposition of advanced kinetic theory of gases. Chapters I, II, IV, and XI – 8 present a self-standing text of the material.

The principal topic of interest in kinetic theory, as it applies to the foundations of the kinetic particle theory of the universe, is the handling of flowing particles. The particles flowing in a given direction have a distribution of varying speeds. To compute the mass flow rate, momentum flow, and energy flow it is necessary to integrate over these speed distributions. The book by Vincenti and Kruger presents this type of analysis.

23. The Grand Unified Theory of Physics

The culmination of all the analysis techniques, the language, the mathematics, and the mechanics is the book *The Grand Unified Theory of Physics* by Joseph M. Brown, published by Basic Research Press, Starkville, MS, 2004. We now present the preface in its entirety.

Preface

This book presents a whole new paradigm for physics. It presents a unified mechanism for deriving all the primary observables in physics. It presents a mechanical model of the neutrino, it shows a mechanism for the fine structure constant and shows why it pervades all of physics, it shows how fundamental particles have a constant value of angular momentum, and it shows the structure of a proton, how its mass, angular momentum, strong nuclear, weak nuclear, and charge fields are produced. A structure of the electron is developed which shows how its mass is held together, how it produces the charge field, and how it produces angular momentum. The book presents the structure of the neutron which gives evidence of how the weak nuclear force functions, and shows the special relativity mechanisms for mass-energy equivalence, mass growth with velocity, matter shortening with velocity, and time dilation. It shows why the mechanism of mass growth of matter with velocity gives matter waves and shows that the waves produce magnetism by the same mechanism that the proton and other charged particles produce their electrostatic field. The book shows that atoms and neutrons produce gravitational fields by a mechanism similar to the breathing sphere model which produces electrostatic fields. The amplitude of the breathing sphere is controlled by, and is equal to, the basic ether particle radius. Further, the same mechanism controlling the breathing sphere amplitude is believed to remove one basic ether particle from a photon for each wave of travel that it executes, which gives the illusion of an expanding universe.

The fine structure constant, 1/137.036, is the ratio of the electromagnetic force to the nuclear force. It also is the velocity of the lowest energy electron orbit in a hydrogen atom in velocity of light units. It pervades all of quantum electrodynamics. However, the number has been a mystery since it was discovered more than seventy years ago. In this model for a grand unified theory of the universe, everything is made up of kinetic particles. The gas of these particles has a root mean square speed that is eight percent larger than the mean speed. We show a model in which the electromagnetic speed (the speed of light) is the difference of these speeds. Also, the same model gives the strong nuclear speed as the background mean speed. Forces generated in a kinetic particle universe, of course, are a function of the square of the speed. The square of the ratio of these two speeds, the root mean square speed less the mean speed to the mean speed is 1/137.109 and thus clearly must be the ratio of the electromagnetic force to the strong nuclear force.

Einstein's special theory of relativity uses a space-time continuum and predicts that as velocity increases, the mass of matter will increase, the length of matter will shorten, and time for processes will increase. Further, the energy content of matter is its mass times the square of the speed of light. The Einstein system is almost universally accepted in science. Many physicists believe it is impossible to derive the theory of relativity observations from classical Newtonian mechanics. In this paper we present a system of absolute space with a separate absolute time, a purely classical (Newtonian) system, from which the above four phenomena are derived. The system used is a kinetic particle system. The model immediately gives the equivalent energy of mass. The model also gives the wave properties of matter in motion.

Magnetism is known to be due to charges in motion. We present a kinetic particle mechanism which produces the electrostatic force, produces the deBroglie wave property of matter, and shows that the deBroglie wave generates the same mechanism which produces the electrostatic force to produce the magnetic force.

The model of the proton structure and its formation, which we present, leads into a hypothesized structure for the electron. With this structure of the electron the structure of the neutron is indicated. We thus present structures for the most basic assemblage of particles, which is the neutrino, and we derive structures for the proton, electron, and neutron.

The mechanism producing gravity is similar to that producing electromagnetism. When the electron is formed the portion of its structure producing the electrostatic field matches the proton electrostatic field except always in the opposite directions. The result is that the flows are matched except for the diameter of a basic ether particle. So, the two fields move with a half amplitude equal the ether particle radius with respect to each other. This then produces the gravitational field. Quantum electrodynamic effects are the result of each elementary matter charge particle consisting of a discrete mass which orbits at the speed of light and produces waves in the background. Matter in motion has a wave path as a result of being accelerated by an eccentric mass. A photon is a narrow "string" of mass which extends completely over one wave length. The particles in the waves have velocities with magnitude and direction. A wave function ψ is a complex number and can be used to describe the expected velocity (magnitude and direction) of a matter or radiation particle. The function ψ is called the probability amplitude. When an event occurs with two possible paths, 1 and 2 then $|\psi_1 + \psi_2|^2$ gives the probability of the combined event. This is the basis of the derivation of the Schroedinger equation. We illustrate the use of probability amplitudes in the analysis of the partial reflection of light and in diffraction gratings.

All of these results taken together can only lead to the conclusion that the universe is made up of Newtonian kinetic particles.

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The Grand Unified Theory of Physics

1. The Postulates

Are there fundamental postulates from which all of physics can be derived? If so, what are these postulates? We have discovered that everything in the universe can be constructed of one type small elastic particle. Yes, just one type particle makes all other particles of matter, makes light and all other radiation, and even makes the elusive neutrinos. We call the basic particle the brutino. The name brutino means tiny brute, since it is very small and is the brute that makes everything. The postulates are:

1. Space is three dimensional.
2. One type of particle makes everything
3. The particle is smooth, elastic, moves, and collides with other particles.

Our first significant discovery toward developing this theory was the discovery of the mechanism of the mysterious fine structure constant.

24. Chemistry

The book *Essentials of General, Organic, and Biological Chemistry* by Drew H. Wolfe, published by McGraw-Hill Company, NY 1986, presents knowledge of chemistry required for a rudimentary understanding of biology.

25. The Biological Cell

The book *The Cell, A Molecular Approach* 4th edition, by Geoffrey M. Cooper and Robert E. Hausman (ISBN-13 978-0-87893-219-1), published by Sinauer Associates, Inc., Sunderland, Massachusetts 2001, presents the basic workings of the biological cell.

26. Biological Aging

The principal remaining unknown about biological cells is what causes aging. In the book *The Chemistry and Mechanics of Human Aging* by Joseph M. Brown, published by Basic Research Press, Starkville, MS 2008, a new theory of aging is presented. Following is an excerpt from the introduction to this book.

All humans age and die. In recent recorded history no human has lived beyond 139 years and it is extremely rare for an individual to live 110 years. The human organism is made up of some 10 billion cells. A portion of these cells divide several times then continue to live indefinitely. However, a large portion of human cells continue to divide throughout the life of the person. As they continue dividing the cell lines begin degrading and eventually die. As a consequence of the degradation and death of cells the complete organism dies. This book investigates the root cause of human aging at the molecular level. A mechanism producing aging is proposed and research is suggested for verifying the proposed cause and for finding a cure. The mechanism consists of chemical forces producing torsional elastic strains at each division in the human genetic material which in turn produces a different chemical constituency of cells and this is manifested as aging.

27. Some Recent Papers by Joseph M. Brown

The foregoing books present a relatively complete development of the fundamentals of a unified theory of physical science and present a possible solution to the most significant remaining problem in the biological sciences. In the book "Some Recent Papers" Dr. Brown presents some recent extensions to the physical science theory:

a. The Human Cosmos

b. Why the Speed of Light Appears to be the Same in All Reference Frames and How to Find the Rest Frame

c. Heat Radiation, Superconductance, and Working with Photons

d. The Kinetic Particle Theory of the Fine Structure Constant

Two of the recent papers deal with the foundations of social science:

e. The Human Soul

f. A Possible Mechanism of Decision Making

28. Epilog

The foregoing material gives an overview of the Unified Science Theory Program of the Basic Research Company. The work on the derivation of language and the foundations of mathematics is believed to be "on the right track." However, the work is untested. The physical theory, as reported in *The Grand Unified Theory of Physics* is quite a departure from current physical thinking but there are so many confirmations of the theory that it almost certainly is correct. The theory of aging as reported in *The Chemistry and Mechanics of Human Aging* may be correct and it should be of interest to aging research scientists. The existence of something beyond physical science as proposed in "The Human Soul" paper is quite tenuous.

Notwithstanding all the shortcomings listed above, it is felt that the scientific material presented here is a "good start" for a program that should be continually extended and refined.