What is the Difference Between Energy and Kinetic Energy?

The energy of matter is mc^2 where m is the mass of the matter and c is the velocity of light. We consider an example of a Newtonian system accelerating a mass m from zero to velocity v, where $v \ll c$. The energy given up by the Newtonian system is mv^2. We also show that the work done by the accelerating system is half the energy given up by the accelerating system.

We know that the energy of matter is mc^2, i.e., its mass times the square of velocity. We say that a ball of mass m translating at velocity v has a kinetic energy of $\frac{1}{2}mv^2$. But, isn't its energy mv^2? We think so. How do we reconcile the fact that the work, i.e. energy expended, which is $\int F\,dx$ required to bring the energy of the ball up to mv^2, when only half that amount of work is required?

Let us illustrate out dilemma with the following example. Let F be constant. Now we write Newton's equation, assuming non-relativistic velocities

$$Fx_1 = \int_0^{x_1} ma\,dx = m\int \frac{dv}{dt}dx = m\int \frac{dv}{dx}\frac{dx}{dt}dx \qquad (1)$$

$$= m\int_0^{v_1} v\,dv = \frac{mv_1^2}{2}$$

where x_1 is the accerlating distance, m is the ball mass, a is the acceleration, v is the ball velocity, and v_1 is the velocity at the end of the acceleration. The energy input Fx_1 is not equal to the ball energy change—it is only half the change of the ball's energy. How can that be? Is the conservation of energy law violated? Let us look at a more complete description of the acceleration process.

To examine the energy involved in the accelerating process, let us consider the acceleration of one elastic ball produced by the repeated impacts by other identical elastic balls. Let us number the impacts from 1 to N and let their velocities be $v_1, v_2, \ldots, v_N$. The individual masses of the impactors and impactees are all m. Let the impactee initially be at rest. Further, let the impact velocities all be central impacts and let all velocities be parallel to the impactee's velocity.[1] Consider the first impact by particle 1. The velocity v_1 is simply transferred to the impactee and the impactor's post-impact velocity is zero. The first impact removes energy mv_1^2 and transfers it to the impactee. The momentum added to the impactee is mv_1. Let the second impactor have velocity $2v_1$. Its post-impact velocity will be v_1 and the impactee's velocity will be $2v_1$. The momentum imparted will be mv_1, the pre-impact energy of the impactor will be $m(2v_1)^2$, and the post impact energy will be mv_1^2, or a loss of $4mv_1^2 - mv_1^2 = 3mv_1^2$. The pre-impact energy of the impactee was mv_1^2 and the post-impact energy of the impactee was mv_1^2 and post-impact energy of the impactee is $m(2v_1)^2 = 4mv_1^2$. The impactee received an energy of $3mv_1^2$ in this impact. Let us summarize the result of impacts.

Impact Num.	Impactor		Impactee		Momentum Imparted	Impactor Energy		Impactee Energy	
	Vel. Bef.	Vel. Aft.	Vel. Bef.	Vel. Aft.		Loss	Cumul.	Gain	Cumul.
1	v_1	0	0	v_1	mv_1	mv_1^2	mv_1^2	$m_1v_1^2$	$m_1v_1^2$
2	$2v_1$	v_1	v_1	$2v_1$	mv_1	$3mv_1^2$	$4mv_1^2$	$3m_1v_1^2$	$4m_1v_1^2$
3	$3v_1$	$2v_1$	$2v_1$	$3v_1$	mv_1	$5mv_1^2$	$9mv_1^2$	$5m_1v_1^2$	$9m_1v_1^2$
4	$4v_1$	$3v_1$	$3v_1$	$4v_1$	mv_1	$7mv_1^2$	$16mv_1^2$	$7m_1v_1^2$	$16m_1v_1^2$
…	…	…	…	…	…	…	…	…	…
N	Nv_1	$(N-1)v_1$	$(N-1)v_1$	Nv_1	mv_1	$(2N-1)mv_1^2$	$N^2mv_1^2$	$(2N-1)m_1v_1^2$	$N^2m_1v_1^2$

The spacing of the impactors is l_n and their velocities are v_n. We take the spacing so the time τ between impacts is the same. Now

$$\tau = \frac{l_n}{v_n} \tag{2}$$

[1] This can be possible only if the impactor is swept out of the approach direction of the impactors as soon as the impact is complete.

and

$$l_n = nl_1, \quad v_n = nv_1 \tag{3}$$

Thus

$$\tau = \frac{l_n}{v_n} = \frac{nl_1}{nv_1} = \frac{l_1}{v_1} \tag{4}$$

We see that this makes the impacts occur at equal time intervals. Force is the time rate of momentum imparted. Thus

$$F = \frac{mv_1}{\tau} = \frac{mv_1 v_1}{l_1} = \frac{mv_1^2}{l} \tag{5}$$

We see the force is constant, $\frac{mv_1^2}{l_1}$. The impactor system energy decreases by $N^2 mv_1^2$ and the impactee energy increases by the same amount. Thus, energy is conserved in this acceleration process. The work W done is the force times the distance, which is

$$\begin{aligned} W_N = F \times (dist) &= \frac{mv_1^2}{l}(l_1 + l_2 + l_3 + \cdots + l_N) \\ &= \frac{mv_1^2}{l_1}(l_1 + 2l_1 + 3l_1 + \cdots + Nl_1) \\ &= mv_1^2(1 + 2 + 3 + \cdots + N) \\ &= mv_1^2 \frac{N(N+1)}{2} \end{aligned} \tag{6}$$

Note that

$$E_N = mv_1^2 N^2 \tag{7}$$

Thus

$$\frac{E_N}{W_N} = \frac{2mv_1^2N^2}{mv_1^2N(N+1)} \approx 2 \tag{8}$$

For large N the energy expended is twice the work produced. Similarly the energy received is the same as the energy expended, but twice the work (energy) done on the accelerated particle. From this we have

$$Work\ Applied\ = \frac{Energy\ Added}{2} = \ Kinetic\ Energy \tag{9}$$

Energy is mass times velocity squared, and energy is conserved in this example.

We will now show that as velocity is increased the factor 2 begins decreasing and approaches unity as the velocity approaches the speed of light. Let us first derive the expression for kinetic energy at high velocity. Here we follow pages 34-35 of Blatt [1].

$$KE\ = \int F\,ds = \int \frac{dp}{dt}ds = \int \frac{ds}{dt}dp = \int v \times (v\,dm + m\,dv) \tag{10}$$

where p is the momentum, v the velocity, and

$$m\ = \frac{m_0}{\sqrt{1-\beta^2}} \tag{11}$$

Equation (11) is a relationship obtained from Newtonian mechanics (see pages 98 to 101 of [1]). Multiplying 2 by $\sqrt{1-\beta^2}$ and squaring both sides and multiplying by c^2 gives

$$m^2c^2 - m^2v^2 = m_0^2c^2 \tag{12}$$

Taking differentials

$$2m\,dm\,c^2 - 2m\,dm\,v^2 - 2m^2 v\,dv = 0 \tag{13}$$

Now

$$c^2 dm = v^2 dm + mv\,dv = v(v\,dm + m\,dv) \tag{14}$$

The right side of (14) is the integrand of (11) so

$$KE = \int_{m_0}^{m} c^2\,dm = mc^2 - m_0 c^2 \tag{15}$$

The energy of a particle at high speed is, as always, the mass times the square of its velocity

$$E = mv^2 = \frac{m_0}{\sqrt{1-\beta^2}} v^2 \tag{16}$$

Now, from (15) and (16)

$$\frac{E}{KE} = \frac{mv^2}{mc^2 - m_0 c^2} = \beta^2 \frac{1}{1 - \frac{m_0}{m}} = \beta^2 \frac{1}{1 - \sqrt{1-\beta^2}} \tag{17}$$

We compute the ratio E/KE for several values of β.

β	$\sqrt{1-\beta^2}$	$\frac{1}{1-\sqrt{1-\beta^2}}$	$\frac{E}{KE}$
0.01	0.99995	19999.5	1.99995
0.1	0.9950	199.5	1.995
0.9	0.4359	1.772	1.435
0.99	0.1411	1.164	1.138
0.999	0.0477	1.0468	1.0458
0.9999	0.01414	1.0143	1.0143

This table shows that the relativistic formula gives the same results previously obtained at low velocity and as the velocity approaches the speed of light the ratio approaches unity.

So, what is energy and what is kinetic energy? Energy is simply mass times the square of its velocity. Kinetic energy is the amount of energy associated with work. In our low-velocity example, the background involved in accelerating the particle gave up an energy equal to mv^2. One half of the expended energy was manifested as work (force times distance). The work was $\frac{1}{2}mv^2$. The receiving particle obtained an energy of mv^2, the same as that lost by the background doing the accelerating.

References

1. Blatt, Frank J. *Modern Physics*. ISBN 9780070058774. McGraw-Hill. New York, NY. 1992.
2. Brown, Joseph M. *The Mechanical Theory of Everything*. ISBN 978-0-9712944-9-3. Basic Research Press. Starkville, MS., 2015.